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On Laplacian energy of graphs

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ABSTRACT

Let G be a graph with n vertices and m edges. Also let $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n = 0$ be the eigenvalues of the Laplacian matrix of graph G. The Laplacian energy of the graph G is defined as

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| .$$

In this paper, we present some lower and upper bounds for LE of graph G in terms of n, the number of edges m and the maximum degree Δ. Also we give a Nordhaus–Gaddum-type result for Laplacian energy of graphs. Moreover, we obtain a relation between Laplacian energy and Laplacian-energy-like invariant of graphs.

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$, $|E(G)| = m$. Let d_i be the degree of the vertex v_i for $i = 1, 2, \ldots, n$. The maximum vertex degree is denoted by Δ. Let $\mathbf{A}(G)$ be the $(0, 1)$-adjacency matrix of G and $\mathbf{D}(G)$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $\mathbf{L}(G) = \mathbf{D}(G) - \mathbf{A}(G)$. This matrix has nonnegative eigenvalues $n \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. Denote by $Spec(G) = \{\mu_1, \mu_2, \ldots, \mu_n\}$ the spectrum of $\mathbf{L}(G)$, i.e., the Laplacian spectrum of G. When more than one graph is under consideration, then we write $\mu_i(G)$ instead of μ_i.

As well known [31],

$$\sum_{i=1}^{n} \mu_i = 2m. \tag{1}$$

The motivation for Laplacian energy comes from graph energy [11,12,21]. The Laplacian energy of the graph G is defined as [26]

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| . \tag{2}$$

For its basic properties, including various lower and upper bounds, see [2,33,34,36–38,41,42]. Laplacian graph energy is a broad measure of graph complexity. Song et al. [35] have introduced component-wise Laplacian graph energy, as a complexity measure useful to filter image description hierarchies.

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Let $\sigma\,(1 \le \sigma \le n-1)$ be the largest integer such that

$$\mu_\sigma \ge \frac{2m}{n}. \tag{3}$$

Then from [9,17,18], we have

$$LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right| = 2S_\sigma(G) - \frac{4m\sigma}{n}, \tag{4}$$

where

$$S_\sigma(G) = \sum_{i=1}^{\sigma} \mu_i.$$

For a subset W of $V(G)$, let $G - W$ be the subgraph of G obtained by deleting the vertices of W and the edges incident with them. If $W = \{v_i\}$, then the subgraph $G - W$ will be written as $G - v_i$ for short. For any two nonadjacent vertices v_i and v_j in graph G, we use $G + v_iv_j$ to denote the graph obtained from adding a new edge v_iv_j to graph G. As usual, K_n and $K_{1,\,n-1}$, denote, respectively, the complete graph and the star on n vertices. For other undefined notations and terminology from graph theory, the readers are referred to [1].
The paper is organized as follows. In Section 2, we give a list of some previously known results. In Section 3, we present some lower and upper bounds on Laplacian energy $LE(G)$ of graph G. In Section 4, we give Nordhaus–Gaddum-type result for Laplacian energy of graphs. In Section 5, we obtain a relation between Laplacian energy and Laplacian-energy-like invariant of graphs.

2. Preliminaries

In this section, we shall list some previously known results that will be needed in the next two sections.

Lemma 2.1 (*[15,19]*). *Let A and B be two real symmetric matrices of size n. Then for any* $1 \le k \le n$,

$$\sum_{i=1}^{k} \lambda_i(A+B) \le \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B).$$

Lemma 2.2 (*[3]*). *Let G be a graph on vertex set* $V(G) = \{v_1, v_2, \ldots, v_n\}$. *Then*

$$\mu_1(G) \le \max_{v_iv_j \in E(G)} |N_G(v_i) \cup N_G(v_j)|, \tag{5}$$

where $N_G(v_i)$ *is the neighbor set of vertex* $v_i \in V(G)$ *and* $|X|$ *is the cardinality of the set X. This upper bound for* $\mu_1(G)$ *does not exceed n.*

Lemma 2.3 (*[31]*). *Let G be a graph on n vertices which has at least one edge. Then*

$$\mu_1 \ge \Delta + 1. \tag{6}$$

Moreover, if G is connected, then the equality holds in (6) if and only if $\Delta = n - 1$.

Lemma 2.4 (*[28]*). *Let G be a graph of order n. Then* $\mu_1(G) \le n$ *with equality holding if and only if* $\overline{G}$ *is disconnected, where* $\overline{G}$ *is the complement of the graph G.*

Lemma 2.5 (*[16]*). *Let* $G(\ncong K_n)$ *be a graph of order n. Then* $\mu_{n-1}(G) \le \delta$, *where* δ *is the minimum degree in G.*

Lemma 2.6 (*[31]*). *Let G be a graph with Laplacian spectrum* $\{0 = \mu_n, \mu_{n-1}, \ldots, \mu_2, \mu_1\}$. *Then the Laplacian spectrum of* $\overline{G}$ *is* $\{0, n-\mu_1, n-\mu_2, \ldots, n-\mu_{n-2}, n-\mu_{n-1}\}$, *where* $\overline{G}$ *is the complement of the graph G.*

Lemma 2.7 (*[6]*). *Let G be a connected graph with* $n \ge 3$ *vertices. Then* $\mu_2 = \mu_3 = \cdots = \mu_{n-1}$ *if and only if* $G \cong K_n$ *or* $G \cong K_{1,\,n-1}$ *or* $G \cong K_{\Delta,\,\Delta}$.

Lemma 2.8 (*[5]*). *Let* $G = (V, E)$ *be a graph with a vertex subset* $V' = \{v_1, v_2, \ldots, v_k\}$ *having the same set of neighbors* $\{v_{k+1}, v_{k+2}, \ldots, v_{k+N}\}$, *where* $V = \{v_1, \ldots, v_k, \ldots, v_{k+N}, \ldots, v_n\}$. *Also let* $E^+ = E \cup E'$, *where* $E' \subseteq V' \times V'$. *If* $G' = (V', E')$