Discrete Mathematics 325 (2014) 65–73

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Disjoint path covers in cubes of connected graphs

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ARTICLE INFO

Article history:
Received 22 August 2012
Received in revised form 7 February 2014
Accepted 12 February 2014
Available online 6 March 2014

Keywords:
Disjoint path cover
Strong hamiltonicity
Hamiltonian path
Prescribed edge
Cube of graph

ABSTRACT

Given a graph G, and two vertex sets S and T of size k each, a many-to-many k-disjoint path cover of G joining S and T is a collection of k disjoint paths between S and T that cover every vertex of G. It is classified as *paired* if each vertex of S must be joined to a designated vertex of T, or *unpaired* if there is no such constraint. In this article, we first present a necessary and sufficient condition for the cube of a connected graph to have a paired 2-disjoint path cover. Then, a corresponding condition for the unpaired type of 2-disjoint path cover problem is immediately derived. It is also shown that these results can easily be extended to determine if the cube of a connected graph has a hamiltonian path from a given vertex to another vertex that passes through a prescribed edge.

1. Introduction

1.1. Problem specification

Given an undirected graph G, a *path cover* is a set of paths in G where every vertex in $V(G)$ is covered by at least one path. Of special interest is the *vertex-disjoint path cover*, or simply called *disjoint path cover*, which is one with an additional constraint that every vertex, possibly except for terminal vertices, must belong to one and only one path. The disjoint path cover made of k paths is called the *k-disjoint path cover* (*k-DPC* for short).

Given two disjoint terminal vertex sets $S = \{s_1, s_2, \ldots, s_k\}$ and $T = \{t_1, t_2, \ldots, t_k\}$ of G, each representing k sources and sinks, the *many-to-many k-DPC* is a disjoint path cover each of whose paths joins a pair of source and sink. The disjoint path cover is regarded as *paired* if every source s_i must be matched with a specific sink t_i. On the other hand, it is called *unpaired* if any permutation of sinks may be mapped bijectively to sources. A graph G is called *paired (resp. unpaired) k-coverable* if $2k \leq |V(G)|$ and there always exists a paired (resp. unpaired) k-DPC for any S and T. The k-DPC has two simpler variants. One is the *one-to-many k-DPC*, whose paths join a single source to k distinct sinks. The other is the *one-to-one k-DPC*, whose paths always start from a single source and end up in a single sink.

The existence of a disjoint path cover in a graph is closely related to the concept of vertex connectivity: Menger's theorem states the connectivity of a graph in terms of the number of disjoint paths joining two distinct vertices, whereas the Fan Lemma states the connectivity of a graph in terms of the number of disjoint paths joining a vertex to a set of vertices [2]. Moreover, it can be shown that a graph is k-connected if and only if it has k disjoint paths joining two arbitrary vertex sets of size k each, in which a vertex that belongs to both sets is counted as a valid path. When a graph does not have a disjoint path cover of desired kind, it is natural to consider an augmented graph with higher connectivity. A simple way of increasing the

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http://dx.doi.org/10.1016/j.disc.2014.02.010
0012-365X/

connectivity is to raise a graph to a power: Given a positive integer d, the d-th power G^d of G is defined as a graph with the same vertex set $V(G)$ and the edge set that is augmented in such a way that two vertices of G^d are adjacent if and only if there exists a path of length at most d in G joining them. In particular, the graph G^2 is called the *square* of G, while G^3 is said to be the *cube* of G.

This paper aims to investigate the structures of the cubes of connected graphs in the point of disjoint path covers. First, we show a necessary and sufficient condition for the cube G^3 of a connected graph G with $|V(G)| \geq 4$ to have a paired 2-DPC joining two arbitrary disjoint vertex sets $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$. Then, the corresponding condition for the existence of an unpaired 2-DPC is immediately derived. In addition, we establish a necessary and sufficient condition under which the cube of a connected graph has an s–t hamiltonian path passing through a prescribed edge e for an arbitrary triple of s, t, and e.

1.2. Disjoint path covers

The disjoint path cover problem has been studied for several classes of graphs: hypercubes [6,9,14,16], recursive circulants [18,19,25,26], and hypercube-like graphs [25,26]. The structure of the cubes of connected graphs was investigated with respect to single-source 3-disjoint path covers [24]. The problem was also investigated in view of a full utilization of nodes in interconnection networks [25]. Its intractability was shown that deciding the existence of a one-to-one, one-to-many, or many-to-many k-DPC in a general graph, joining arbitrary sets of sources and sinks, is NP-complete for all $k \geq 1$ [25,26].

The method for finding a disjoint path cover can easily be used for finding a hamiltonian path (or cycle) due to its natural relation to the hamiltonicity of graph. For instance, a hamiltonian path between two distinct vertices in a graph G is in fact a 1-DPC of G joining the vertices. An s–t hamiltonian path in G that passes an arbitrary sequence of k pairwise nonadjacent edges $((x_1, y_1), (x_2, y_2), \dots, (x_k, y_k))$ in the specified order always exists for any distinct s and t with $s \neq x_i, y_i$ and $t \neq x_i, y_i$ $(1 \leq i \leq k)$ if G is paired $(k+1)$-coverable [25]. A simpler, s–t hamiltonian path that passes a prescribed edge (x, y) with $\{s, t\} \cap \{x, y\} = \emptyset$ can also be found by solving the corresponding unpaired or paired 2-DPC problem [26]. While the unpaired version would be easier to tackle than the paired one, the difference is that the direction between x and y in the path may not be enforced through the unpaired 2-DPC. For more discussion on the hamiltonian paths (or cycles) passing through prescribed edges, refer to, for example, [3,8].

1.3. Strong hamiltonian properties

The cube of a connected graph with at least four vertices is 1-hamiltonian, i.e., it is hamiltonian and remains so after the removal of any one vertex, as Chartrand and Kapoor showed [5]. Sekanina [29] and Karaganis [17] independently proved that the cube of a connected graph is hamiltonian-connected. Whether the cube is 1-hamiltonian-connected, i.e., it still remains hamiltonian-connected after the removal of any one vertex, was characterized for trees by Lesniak [21] and for connected graphs by Schaar [28]. Characterizations of connected graphs whose cubes are p-hamiltonian for $p \leq 3$ were also made in [20,28], and strong hamiltonian properties of the cube of a 2-edge connected graph were studied in [23].

On the other hand, the hamiltonicity of the square of a graph was investigated by several researchers. Fleischner proved that the square of every 2-connected graph is hamiltonian [11] (for an alternative proof, refer to [13] or [22]). In fact, the square of a 2-connected graph is both hamiltonian-connected and 1-hamiltonian provided that its order is at least four [4]. These works were followed by several results on the hamiltonicity of the square graphs, in particular, by Abderrezzak et al. [1], Chia et al. [7], and Ekstein [10]. Interesting results on pancyclicity and panconnectivity of the square of a connected graph were given in [12]. For the square of a tree T, Harary and Schwenk showed that T^2 is hamiltonian if and only if T is a caterpillar [15]. Recently, Radoszewski and Rytter proved that T^2 has a hamiltonian path if and only if T is a horsetail [27].

1.4. Fundamental concepts and notation

Before proceeding to the main results, we summarize some fundamental terminologies on the graph connectivity that are frequently used in this article (refer to Fig. 1 for an illustration of them). An edge of a graph G is called a *bridge*, or *cut-edge*, if its removal increases the number of connected components. Trivially, an edge is a bridge if and only if it is not contained in a cycle. A bridge is said to be *nontrivial* if none of its two end-vertices is of degree one. A vertex of G is called a *pure bridge vertex* if each of its incident edges is a nontrivial bridge. Furthermore, a set of three pairwise adjacent vertices, each having a degree of at least three, is called a *pure bridge triangle* if every edge that is incident with exactly one of the triangular vertices is a nontrivial bridge. In addition to these terminologies, we introduce another one:

Definition 1. A set of two adjacent vertices is called a *pure bridge pair* if both vertices are pure bridge vertices.

Next, we present two fundamental theorems about the hamiltonian properties of the cubes of connected graphs that play important roles in deriving our results.

Theorem 1 (*Sekanina [29] and Karaganis [17]*). *The cube of every connected graph is hamiltonian-connected.*