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Note

A note on the number of spanning trees of line digraphs

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ABSTRACT

Let G be a digraph and $\mathcal{L}G$ be its line digraph. Levine gave a formula that relates the number of rooted spanning trees of $\mathcal{L}G$ and that of G, with the restriction that G has no sources. In this note, we show that this restriction can be removed, thus his formula holds for all digraphs.

1. Terminology and notation

Since the discussion is based on the results of Levine [3], some notation from that paper are used in this note for consistency. The digraphs considered here may have loops and multiple edges. For terminology and notation not defined we refer the reader to Bondy and Murty [1].

Let G be a finite digraph. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G respectively. For an edge e of G which directs from a vertex u to a vertex v, u is said to be the *tail* of e, denoted by $t(e) = u$, and v the *head* of e, denoted by $h(e) = v$. The *out-degree* and *in-degree* of vertex v are defined by $\mathrm{outdeg}(v) = \left|\{e : t(e) = v\}\right|$ and $\mathrm{indeg}(v) = \left|\{e : h(e) = v\}\right|$, respectively. G is said to be *k-out-regular* (*k-in-regular*) if $\mathrm{outdeg}(v) = k$ ($\mathrm{indeg}(v) = k$) for every vertex v of G. A *source* of G is a vertex with in-degree 0. Note that if there is a loop e at v ($t(e) = h(e) = v$), then v is not a source.

A *rooted spanning tree* of a digraph G is an oriented spanning tree such that every vertex except one (the *root*) has out-degree 1. The number of rooted spanning trees of G is denoted by $\kappa(G)$. Let $\{x_v\}_{v\in V(G)}$ and $\{x_e\}_{e\in E(G)}$ be the indeterminates on the vertex set and edge set of G, which can be regarded as weights on the vertices and edges respectively. In our discussion, we will assume that $x_v > 0$ for all $v \in V(G)$ and $x_e > 0$ for all $e \in E(G)$. Consider the following two polynomials

$$\kappa^{edge}(G, \mathbf{x}) = \sum_T \prod_{e\in T} x_e \quad \text{and} \quad \kappa^{vertex}(G, \mathbf{x}) = \sum_T \prod_{e\in T} x_{h(e)},$$

where the sums are taken over all the rooted spanning trees of G. It is trivial that $\kappa^{edge}(G, \mathbf{1}) = \kappa^{vertex}(G, \mathbf{1}) = \kappa(G)$, where $\mathbf{1}$ is the vector with 1 as all of its components.

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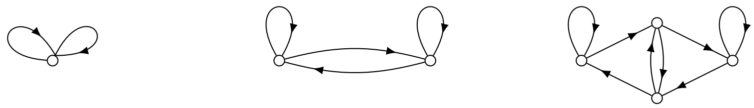

Fig. 1. Digraphs G, $\mathcal{L}G$, and $\mathcal{L}^2G$.

The *line digraph* of G is the digraph $\mathcal{L}G$ with vertex set $V(\mathcal{L}G) = E(G)$ and edge set $E(\mathcal{L}G) = \big\{(e, f) \in E(G) \times E(G) | h(e) = t(f)\big\}$ (see Fig. 1). The vertex weight of $\mathcal{L}G$ is the same as the edge weight of G, i.e., the vertex e in $\mathcal{L}G$ has weight x_e.

For an unweighted digraph G, the iterated line digraph $\mathcal{L}^tG = \mathcal{L}\big(\mathcal{L}^{t-1}G\big)$ is the t-th line digraph of G. The vertex set of $\mathcal{L}^tG$ is denoted by

$$E_t = \Big\{(e_1, e_2, \ldots, e_t) \in E^n \Big| t(e_{i+1}) = h(e_i), i = 1, \ldots, t-1\Big\},$$

in which whose elements are directed walks in G with t edges. This definition of E_t is slightly different from the definition in [3] where the elements of E_t are directed paths in G with t edges.

2. Main results

There is a well-known result which relates the number of spanning trees of a regular (undirected) graph and its line graph (see [2, pp. 218]): If a graph G is k-regular, then

$$\kappa(\mathcal{L}G) = 2^{m-n+1}k^{m-n-1} \cdot \kappa(G).$$

A directed version of this result follows from the following theorem.

Theorem 1 (*Zhang, Zhang and Huang [4]*)**.** *Let G be a digraph with order n. If G is both k-out-regular and k-in-regular, then*

$$\kappa(\mathcal{L}^tG) = k^{(k^t-1)n} \cdot \kappa(G).$$

By setting $t = 1$ in Theorem 1, for digraphs which are both k-out-regular and k-in-regular there holds

$$\kappa(\mathcal{L}G) = k^{m-n} \cdot \kappa(G).$$

In 2011, Levine [3] gave a formula that relates $\kappa^{vertex}(\mathcal{L}G, \mathbf{x})$ and $\kappa^{edge}(G, \mathbf{x})$.

Theorem 2 (*Levine [3]*)**.** *Let $G = (V, E)$ be a finite digraph with no sources. Then*

$$\kappa^{vertex}(\mathcal{L}G, \mathbf{x}) = \kappa^{edge}(G, \mathbf{x}) \cdot \prod_{v\in V}\Bigg(\sum_{t(e)=v} x_e\Bigg)^{\mathrm{indeg}(v)-1}.$$

Here we find that the restriction G has no sources can be removed.

Theorem 3. *Let $G = (V, E)$ be a finite digraph. Then*

$$\kappa^{vertex}(\mathcal{L}G, \mathbf{x}) = \kappa^{edge}(G, \mathbf{x}) \cdot \prod_{v\in V}\Bigg(\sum_{t(e)=v} x_e\Bigg)^{\mathrm{indeg}(v)-1}.$$

Proof. If G has no sources, the result follows from Theorem 2. Now we assume that G has a nonempty set of sources $S = \{s_1, s_2, \ldots, s_t\}$. By adding a weighted loop l_i at each s_i for $1 \le i \le s$, there obtains another digraph G^* (see Fig. 2). It is trivial that

$$\kappa^{edge}(G^*, \mathbf{x}) = \kappa^{edge}(G, \mathbf{x}).$$

The out-neighbours of l_i in $\mathcal{L}G^*$ are exactly the out-going edges of v_i in G, and l_i has no in-neighbours in $\mathcal{L}G^*$, for all $1 \le i \le t$. There holds

$$\kappa^{vertex}(\mathcal{L}G^*, \mathbf{x}) = \kappa^{vertex}\big(\mathcal{L}(G^* - l_i), \mathbf{x}\big) \cdot \sum_{\substack{e\in E(G)\\ t(e)=s_i}} x_e, \quad 1 \le i \le t.$$

Moreover, note that $\mathcal{L}G = \mathcal{L}G^* - \{l_1, l_2, \ldots, l_t\}$, and $\{l_1, l_2, \ldots, l_t\}$ forms an independent set of $\mathcal{L}G^*$. There follows

$$\kappa^{vertex}(\mathcal{L}G^*, \mathbf{x}) = \kappa^{vertex}(\mathcal{L}G, \mathbf{x}) \cdot \prod_{v\in S}\Bigg(\sum_{\substack{e\in E(G)\\ t(e)=v}} x_e\Bigg). \tag{1}$$