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Maximal harmonic group actions on finite graphs

Scott Corry

Department of Mathematics, Lawrence University, 711 E. Boldt Way – SPC 24, Appleton, WI 54911, USA

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ABSTRACT

This paper studies groups of maximal size acting harmonically on a finite graph. Our main result states that these maximal graph groups are exactly the finite quotients of the modular group $\Gamma = \langle x, y \mid x^2 = y^3 = 1 \rangle$ of size at least 6. This characterization may be viewed as a discrete analogue of the description of Hurwitz groups as finite quotients of the (2, 3, 7)-triangle group in the context of holomorphic group actions on Riemann surfaces. In fact, as an immediate consequence of our result, every Hurwitz group is a maximal graph group, and the final section of the paper establishes a direct connection between maximal graphs and Hurwitz surfaces via the theory of combinatorial maps.

1. Introduction

Many recent papers have explored analogies between Riemann surfaces and finite graphs (e.g. [2–4,6–8,14,15,17]). Inspired by the Accola–Maclachlan [1,21] and Hurwitz [18] genus bounds for holomorphic group actions on compact Riemann surfaces, we introduced harmonic group actions on finite graphs in [14], and established sharp linear genus bounds for the maximal size of such actions. As noted in the introduction to [14], it is an interesting problem to characterize the groups and graphs that achieve the upper bound $6(g-1)$. Such maximal groups and graphs may be viewed as graph-theoretic analogues of Hurwitz groups and surfaces—those compact Riemann surfaces $\mathcal{S}$ of genus $g \geq 2$ such that $\mathrm{Aut}(\mathcal{S})$ has maximal size $84(g-1)$. This paper provides a description of the maximal graphs and groups (Theorem 1 and Proposition 12), while also establishing connections between the recent theory of harmonic group actions and the well-studied topics of trivalent symmetric graphs and regular combinatorial maps.

The investigation of Hurwitz groups has been a rich and active area of research, and much is known about their classification including a complete analysis of the 26 sporadic simple groups: 12 of them (including the Monster!) are Hurwitz, while the other 14 are not (see [9,10] for an overview). One starting point for work on Hurwitz groups is the following generation result: a finite group G is a Hurwitz group if and only if it is a non-trivial quotient of the (2, 3, 7)-triangle group Δ with presentation

$$\Delta = \left\langle x, y \mid x^2 = y^3 = (xy)^7 = 1 \right\rangle.$$

That is: the Hurwitz groups are exactly the finite groups generated by an element of order 2 and an element of order 3 such that their product has order 7. The connection between the abstract group Δ and Hurwitz groups comes from the fact that Hurwitz surfaces arise as branched covers of the thrice-punctured Riemann sphere with special ramification. Such covers are nicely classified by the fundamental group of the punctured sphere, which is a free group on two generators.

The main result of this paper is an analogous generation result for *maximal graph groups*—those finite groups of size $6(g-1)$ that act harmonically on a finite graph of genus $g \geq 2$:

E-mail address: scott.corry@lawrence.edu.

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Theorem 1. *A finite group G is a maximal graph group if and only if* $|G| \geq 6$ *and G is a quotient of the modular group* Γ *with presentation*

$$\Gamma = \langle x, y \mid x^2 = y^3 = 1 \rangle .$$

That is: the maximal graph groups are exactly the finite groups generated by an element of order 2 and an element of order 3. As an immediate corollary, we have:

Corollary 2. *Every Hurwitz group is a maximal graph group.*

As in the case of Hurwitz groups, the connection between the modular group Γ and maximal graph groups comes from the fact that maximal graphs occur as harmonic branched covers of trees (genus 0 graphs) with special ramification (Proposition 11). In order to classify such covers in general, we developed a harmonic Galois theory for finite graphs in [15], and the resulting concrete description of harmonic branched covers is the main tool used in the proof of Theorem 1, which we present in Section 3. The proof of Theorem 1 leads immediately to Proposition 12, which provides a close connection between maximal graphs and trivalent symmetric graphs of type 1' studied by Djoković and Miller in [16].

The relation between Riemann surfaces and finite graphs explored in this paper is largely analogical, rather than arising from a precise correspondence. However, there are a variety of direct connections between Riemann surfaces (and more generally algebraic curves) and finite graphs (see e.g. [2,5–7,19]). Of particular interest for us is a portion of the well-established theory of combinatorial maps, whereby the specification of a cyclic ordering of the edges incident to each vertex of a finite graph determines a 2-cell embedding of the graph in a compact Riemann surface. In the final section of this paper, we show that our theory meshes well with this construction in the following sense: if G is a maximal graph group, then (by Proposition 12) G acts maximally on a trivalent graph Y_0. Moreover, the G-action endows Y_0 with a cyclic ordering of the three edges at each vertex, and G acts as a group of holomorphic automorphisms of the corresponding Riemann surface. Moreover, if G is actually a Hurwitz group, then the resulting surface is a Hurwitz surface with automorphism group G.

2. Harmonic group actions

In this section, we briefly review some of the definitions and results from [4,14,15]. To begin, by a *graph* we mean a finite multi-graph without loop edges: two vertices may be connected by multiple edges, but no vertex has an edge to itself. We denote the (finite) vertex-set of a graph X by $V(X)$, and the (finite) edge-set by $E(X)$. For a vertex $x \in V(X)$, we write $x(1)$ for the subgraph of X induced by the edges incident to x:

$$V(x(1)) := \{x\} \cup \{w \in V(X) \mid w \text{ is adjacent to } x\}$$
$$E(x(1)) := \{e \in E(X) \mid e \text{ is incident to } x\}.$$

The *genus*[1] of a connected graph X is the rank of its first Betti homology group: $g(X) := |E(X)| - |V(X)| + 1$.

Definition 3. A *morphism of graphs* $\phi : Y \to X$ is a function $\phi : V(Y) \cup E(Y) \to V(X) \cup E(X)$ mapping vertices to vertices and such that for each edge $e \in E(Y)$ with endpoints $y_1 \neq y_2$, either $\phi(e) \in E(X)$ has endpoints $\phi(y_1) \neq \phi(y_2)$, or $\phi(e) = \phi(y_1) = \phi(y_2) \in V(X)$. In the latter case, we say that the edge e is ϕ-*vertical*. The morphism ϕ is *degenerate* at $y \in V(Y)$ if $\phi(y(1)) = \{\phi(y)\}$, i.e. if ϕ collapses a neighborhood of y to a vertex of X. The morphism ϕ is *harmonic* if for all vertices $y \in V(Y)$, the quantity $|\phi^{-1}(e') \cap y(1)|$ is independent of the choice of edge $e' \in E(\phi(y)(1))$. See Fig. 1.

Definition 4. Let $\phi : Y \to X$ be a harmonic morphism between graphs, with X connected. If $|V(X)| > 1$ (i.e. if X is not the point graph $\star$), then the *degree* of the harmonic morphism ϕ is the number of pre-images in Y of any edge of X (this is well-defined by [4], Lemma 2.4). If $X = \star$ is the point graph, then the *degree* of ϕ is defined to be $|V(Y)|$, the number of vertices of Y.

Definition 5. Suppose that $G \leq \mathrm{Aut}(Y)$ is a (necessarily finite) group of automorphisms of the graph Y, so that we have a left action $G \times Y \to Y$ of G on Y. We say that (G, Y) is a *faithful group action* if the stabilizer of each connected component of Y acts faithfully on that component. Note that this condition is automatic if Y is connected.

Given a faithful group action (G, Y), we denote by $G\backslash Y$ the quotient graph with vertex-set $V(G\backslash Y) = G\backslash V(Y)$, and edge-set

$$E(G\backslash Y) = G\backslash E(Y) - \{Ge \mid e \text{ has endpoints } y_1, y_2 \text{ and } Gy_1 = Gy_2\}.$$

Thus, the vertices and edges of $G\backslash Y$ are the left G-orbits of the vertices and edges of Y, with any loop edges removed. There is a natural morphism $\phi_G : Y \to G\backslash Y$ sending each vertex and edge to its G-orbit, and such that edges of Y with endpoints in the same G-orbit are ϕ_G-vertical. As demonstrated in Fig. 2, the quotient morphism ϕ_G is not necessarily harmonic, which motivates the following definition.

[1] In graph theory, the term "genus" usually refers to the minimal genus of an orientable surface into which a graph may be embedded, while the first Betti number of the graph is called the *cyclomatic number*. But following [3], we will refer to the quantity $g(X)$ as the genus, because in the theory of divisors on the graph X, it plays a role analogous to the genus of a Riemann surface.