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2-factors with bounded number of components in claw-free graphs

Kenta Ozeki [a,b,*], Zdeněk Ryjáček [c,1], Kiyoshi Yoshimoto [d]

[a] National Institute of Informatics, 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan
[b] JST, ERATO, Kawarabayashi Large Graph Project, Japan
[c] Department of Mathematics, University of West Bohemia, P.O. Box 314, 306 14 Pilsen, Czech Republic
[d] Department of Mathematics, College of Science and Technology, Nihon University, Tokyo 101-8308, Japan

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ABSTRACT

In this paper, we show that every 3-connected claw-free graph G has a 2-factor having at most $\max\left\{\frac{2}{5}(\alpha+1), 1\right\}$ cycles, where α is the independence number of G. As a corollary of this result, we also prove that every 3-connected claw-free graph G has a 2-factor with at most $\left(\frac{4|G|}{5(\delta+2)}+\frac{2}{5}\right)$ cycles, where δ is the minimum degree of G. This is an extension of a known result on the number of cycles of a 2-factor in 3-connected claw-free graphs.

1. Introduction

A well-known conjecture by Matthews and Sumner [17] states that every 4-connected claw-free graph is Hamiltonian. Recall that a graph is *claw-free* if it has no claw $K_{1,3}$ as an induced subgraph. Thomassen [20] also posed the following conjecture: every 4-connected line graph is Hamiltonian. Note that Ryjáček [18] showed that these two conjectures are equivalent, using a closure technique. These two conjectures have attracted much attention during the last more than 25 years, but they are still open.

To attack these conjectures, some researchers have considered the Hamiltonicity of claw-free graphs with high connectivity conditions. In fact, Zhan [22], and independently Jackson [12] proved that Thomassen's conjecture is true for 7-connected line graphs. Recently, Kaiser and Vrána [15] improved this result by showing that every 5-connected claw-free graph with minimum degree at least six is Hamiltonian. Like these, several researchers have attacked these conjectures in claw-free graphs with high connectivity. See for example [11,23].

On the other hand, it is also natural to ask what happens when we consider claw-free graphs with low connectivity. Although it is known that there exist infinitely many 3-connected claw-free graphs (also line graphs) having no Hamiltonian cycles, we would like to find some "good" structures which have some properties close to Hamiltonian cycles in such graphs. The main target of this paper is a 2-*factor* with a bounded number of components. (See the survey [7] for other "good" structures.)

Recall that a 2-factor of a graph is a spanning subgraph in which all vertices have degree two. A Hamiltonian cycle of a graph is actually a 2-factor with exactly one component. In this sense, the fewer components a 2-factor has, the closer to a

* Corresponding author at: National Institute of Informatics, 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan.
E-mail addresses: ozeki@nii.ac.jp (K. Ozeki), ryjacek@kma.zcu.cz (Z. Ryjáček), yosimoto@math.cst.nihon-u.ac.jp (K. Yoshimoto).

1 Institute for Theoretical Computer Science (ITI), Charles University, Prague, Czech Republic.

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Hamiltonian cycle it is. Choudum and Paulraj [4], and independently Egawa and Ota [5] proved that if the minimum degree of a claw-free graph G is at least four, then G has a 2-factor (without considering the number of components). Yoshimoto [21] showed that if G is a 2-connected claw-free graph with minimum degree at least three (specifically, if G is 3-connected), then G has a 2-factor.

Now we consider a 2-factor with bounded number of cycles in claw-free graphs. Faudree, Favaron, Flandrin, Li and Liu [6] showed that a claw-free graph with minimum degree $\delta \geq 4$ has a 2-factor with at most $\frac{6|G|}{\delta+2} - 1$ cycles. Gould and Jacobson [10] improved this result for a claw-free graph with large minimum degree; a claw-free graph with minimum degree $\delta \geq (4|G|)^{\frac{2}{3}}$ has a 2-factor with at most $\lceil \frac{|G|}{\delta} \rceil$ cycles. Recently, Broersma, Paulusma and Yoshimoto showed the following result.

Theorem 1 (*Broersma, Paulusma and Yoshimoto [1]*)**.** *Every claw-free graph G with minimum degree $\delta \geq 4$ has a 2-factor with at most*

$$\max\left\{\frac{|G|-3}{\delta-1}, 1\right\} \textit{ cycles.}$$

Also Yoshimoto [21] showed that the coefficient $\frac{1}{\delta-1}$ of $|G|$ is almost best possible.

Now we consider 2-connected or 3-connected claw-free graphs. Jackson and Yoshimoto [13] showed that every 2-connected claw-free graph G with minimum degree at least four has a 2-factor with at most $\frac{|G|+1}{4}$ cycles, and moreover, with at most $\frac{2|G|}{15}$ cycles if G is 3-connected. Čada, Chiba and Yoshimoto [2] proved that every 2-connected claw-free graph G with minimum degree $\delta \geq 4$ has a 2-factor in which every cycle has the length at least δ. This result implies the existence of a 2-factor with at most $\frac{|G|}{\delta}$ cycles in a 2-connected claw-free graph G.

On the other hand, Kužel, Ozeki and Yoshimoto [16] focused on a relationship between a 2-factor and maximum independent sets in a graph, and showed the following:

Theorem 2 (*Kužel, Ozeki and Yoshimoto [16]*)**.** *For every maximum independent set S in a 2-connected claw-free graph G with minimum degree at least three, G has a 2-factor in which each cycle contains at least one vertex in S, and moreover, at least two vertices in S if G is 3-connected.*

As a direct corollary of Theorem 2, we obtain that every 3-connected claw-free graph G has a 2-factor with at most $\alpha/2$ cycles, where α is the independence number of G. Note that for every claw-free graph G, we have that $\alpha \leq \frac{2|G|}{\delta+2}$, where α is the independence number and δ is the minimum degree of G, respectively. (See for example, Fact 8 in [8].) Therefore, the result of Kužel et al. implies the following corollary.

Theorem 3 (*Kužel, Ozeki and Yoshimoto [16]*)**.** *Every 3-connected claw-free graph G with minimum degree δ has a 2-factor with at most*

$$\max\left\{\frac{|G|}{\delta+2}, 1\right\} \textit{ cycles.}$$

In this paper, we show the following result, which means that if we do not specify a maximum independent set, for 3-connected claw-free graphs, we can find a 2-factor with fewer cycles than the one obtained by Theorem 2.

Theorem 4. *Every 3-connected claw-free graph with independence number α has a 2-factor with at most*

$$\max\left\{\frac{2}{5}(\alpha+1), 1\right\} \textit{ cycles.}$$

We do not know whether the coefficient $\frac{2}{5}$ of α in Theorem 4 is best possible or not. However, in Section 3, we show two examples to discuss sharpness of the result. By the same argument as above, Theorem 4 implies the following corollary.

Corollary 5. *Every 3-connected claw-free graph G with minimum degree δ has a 2-factor with at most*

$$\left(\frac{4|G|}{5(\delta+2)} + \frac{2}{5}\right) \textit{ cycles.}$$

In Corollary 5, we decrease the coefficient of $|G|$ in Theorem 3. This is the first result that guarantees, in a 3-connected claw-free graph, the existence of a 2-factor having number of cycles with coefficient of $|G|/\delta$ less than 1.

In the next section, we give two statements (Theorems 6 and 7), that are equivalent to Theorem 4. After discussing sharpness of Theorem 4 in Section 3, we show some lemmas in Sections 4 and 5. In Section 6, we prove Theorem 7.