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Labeled 2-packings of trees ☆

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ABSTRACT

Graph packing generally deals with unlabeled graphs. In Duchêne et al. (2013), the authors introduced a new variant of the graph packing problem, called *labeled packing* of a graph. In the current paper, we present several results about the labeled packing number of trees. Exact values are given in the cases of paths and caterpillars. For a general tree, a lower bound is given thanks to the introduction of the concept of fixed-point free labeled packing.

1. Context and definitions

1.1. Graph theoretical definitions

All graphs considered in this paper are finite, undirected, without loops or multiple edges. An end-vertex in a graph is a vertex of degree 1. For a graph G, we will use $V(G)$ and $E(G)$ to denote its vertex and edge sets respectively. Given $V' \subset V(G)$, the subgraph $G[V']$ denotes the subgraph of G induced by V', i.e., $E(G[V'])$ contains all the edges of E having both end-vertices in V'. When a graph G has order n and size m, we say that G is an (n, m)-graph.

An *independent set* of G is a set $X \subseteq V(G)$ such that no two vertices in X are adjacent. An independent set is *maximal* if no independent set properly contains it. An independent set of maximum cardinality is a *maximum independent set*. For undefined terms, we refer the reader to [3].

A *permutation* σ is a one-to-one mapping of $\{1, \ldots, n\}$ into itself. We say that a permutation σ is *fixed-point-free* if $\sigma(x) \neq x$ for all x.

1.2. The graph packing problem

The graph packing problem was introduced by Bollobás and Eldridge [2] and Sauer and Spencer [10] in the late 1970s. Let $G_1, \ldots, G_k$ be k graphs of order n. We say that there is a *packing* of $G_1, \ldots, G_k$ (into the complete graph K_n) if there exist permutations $\sigma_i : V(G_i) \longrightarrow V(K_n)$, where $1 \leq i \leq k$, such that $\sigma_i^*(E(G_i)) \cap \sigma_j^*(E(G_j)) = \emptyset$ for $i \neq j$, and here the map $\sigma_i^* : E(G_i) \longrightarrow E(K_n)$ is the one induced by σ_i.

A packing of k copies of the same graph G will be called a *k-packing of* G. In other words, a 2-packing of a graph G is a permutation σ on $V(G)$ such that if an edge vu belongs to $E(G)$, then $\sigma(v)\sigma(u)$ does not belong to $E(G)$.

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Graph packing is a well known field of graph theory that has been considerably investigated in the literature. An overview of the domain can be found in the survey papers of Wozniak [12] and Yap [13]. In particular, the question of the existence of a 2-packing of a given graph has received a great attention. In [4], a full characterization of all the 2-packable $(n, n-1)$-graphs is given. Note that stars are the only connected $(n, n-1)$-graphs that are not 2-packable. A similar result about 2-packable (n, n)-graphs can be found in [6]. Other papers about packing of different trees into K_n can be found in [7–9].

In collaboration with R.J. Nowakowski, we have recently introduced a labeled version of graph packing [5]. Roughly speaking, it consists of a graph packing that preserves the labels of the vertices. We give below a more formal definition of this problem.

Definition 1 *([5])***.** Given a positive integer p, let G be a graph of order n and f be a mapping from $V(G)$ to the set $\{1, \ldots, p\}$. The mapping f is called a *p-labeled k-packing* of G into K_n if there exist permutations $\sigma_i : V(G) \longrightarrow V(K_n)$ for $1 \leq i \leq k$ such that:

(1) $\sigma_i^*(E(G)) \cap \sigma_j^*(E(G)) = \emptyset$ for all $i \neq j$.
(2) For every vertex v of G, we have $f(v) = f(\sigma_1(v)) = f(\sigma_2(v)) = \cdots = f(\sigma_k(v))$.

The maximum positive integer p for which G admits a p-labeled k-packing is called the *labeled k-packing number* of G and is denoted by $\lambda_k(G)$. Throughout this paper, a labeled 2-packing of G will be denoted by a pair (f, σ).

Remark 2. The existence of a k-packing of G is a necessary condition for the existence of p-labeled k-packing of G. It suffices to choose $p = 1$.

In [5], labeled packing of cycles is studied. More precisely, the labeled k-packing number of cycles is exactly determined for $k = 2$. For larger k, the problem is almost solved, except for cycles C_n with $2k + 1 \leq n \leq 4k - 4$.

2. Preliminary results

2.1. General results about labeled 2-packings of graphs

The current paper will deal with labeled 2-packings of trees. First, we give some useful results about labeled 2-packings in a more general context. In particular, the following lemma is a direct consequence of Definition 1.

Lemma 3. *Let G be a graph of order n, and let H be a spanning subgraph of G. If there exists a 2-packing of G into K_n, then*

$$\lambda_2(H) \geq \lambda_2(G).$$

Proof. Indeed, since $H(E) \subseteq G(E)$, any labeled 2-packing of G is also a labeled 2-packing of H. □

The following theorem, which was proved in [5], gives an upper bound for the labeled 2-packing number of a general graph.

Theorem 4 *([5])***.** *Let G be a graph of order n and let I be a maximum independent set of G. If there exists a 2-packing of G into K_n, then*

$$\lambda_2(G) \leq |I| + \left\lfloor \frac{n - |I|}{2} \right\rfloor.$$

To understand this result, consider a maximum independent set I of G and a maximum labeled 2-packing (f, σ). The upper bound of Theorem 4 will be reached if one can set the vertices of I as fixed points of σ, together with finding a fixed-point-free $\lfloor \frac{|V(G) \setminus I|}{2} \rfloor$-labeled 2-packing of the subgraph induced by $V(G) \setminus I$. For example, let us consider the caterpillar T of Fig. 1(a). From Theorem 4, we have $\lambda_2(T) \leq 10$. To achieve this bound it is necessary to find a 3-labeled fixed-point-free 2-packing of the central path of T (Fig. 1(b)).

Remark 5. In any labeled 2-packing of a graph G induced by a permutation σ, the vertices of every cycle in σ must have the same label. Therefore, the labeled 2-packing number of a graph is the maximum number of cycles in a 2-packing of G.

2.2. Labeled fixed-point-free 2-packings

The above considerations lead us to introduce a new packing problem called *labeled fixed-point-free 2-packing* of graphs:

Definition 6. Let (f, σ) be a p-labeled 2-packing of a given graph G. We say that f is a *p-labeled fixed-point-free 2-packing* if σ is a fixed-point-free permutation.

The maximum positive integer p for which G admits a labeled fixed-point-free 2-packing of G will be called the *labeled fixed-point-free 2-packing number* of G and denoted by $\alpha_2(G)$.

Remark 7. Note that Lemma 3 is also valid for the fixed-point-free 2-packing number.