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Describing short paths in plane graphs of girth at least 5

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ABSTRACT

We prove that every connected plane graph of given girth g and minimum degree at least 2 contains an edge whose degrees are bounded from above by one of the pairs $(2, 5)$ or $(3, 3)$ if $g = 5$, by pair $(2, 5)$ if $g = 6$, by pair $(2, 3)$ if $g \in \{7, 8, 9, 10\}$, and by pair $(2, 2)$ if $g \geq 11$. Further we prove that every connected plane graph of given girth g and minimum degree at least 2 has a path on three vertices whose degrees are bounded from above by one of the triplets $(2, \infty, 2)$, $(2, 2, 6)$, $(2, 3, 5)$, $(2, 4, 4)$, or $(3, 3, 3)$ if $g = 5$, by one of the triplets $(2, 2, \infty)$, $(2, 3, 5)$, $(2, 4, 3)$, or $(2, 5, 2)$ if $g = 6$, by one of the triplets $(2, 2, 6)$, $(2, 3, 3)$, or $(2, 4, 2)$ if $g = 7$, by one of the triplets $(2, 2, 5)$ or $(2, 3, 3)$ if $g \in \{8, 9\}$, by one of the triplets $(2, 2, 3)$ or $(2, 3, 2)$ if $g \geq 10$, and by the triplet $(2, 2, 2)$ if $g \geq 16$.

1. Introduction

A normal plane map is a plane graph in which loops and multiple edges are allowed, but the degree of each vertex and each face is at least three. In this paper we investigate 2-connected plane graphs in which loops and multiple edges are not allowed. We use a standard graph theory terminology according to the book [2]. However we recall some more frequent notions. The facial walk of a face α is the shortest closed walk containing all edges incident with α. The degree of a vertex v or a face α, that is the number of edges incident with v or the length of the facial walk of α, is denoted by $\deg(v)$ or $\deg(\alpha)$, respectively. A k-vertex (k-face) is a vertex v (face α) with $\deg(v) = k$ ($\deg(\alpha) = k$). By k^+ or k^- we denote any integer not smaller or not greater than k, respectively. Hence, a k^+-vertex v (k^+-face α) satisfies $\deg(v) \geq k$ ($\deg(\alpha) \geq k$) and k^--vertex v (k^--face α) satisfies $\deg(v) \leq k$ ($\deg(\alpha) \leq k$). An edge uv is of the type (i, j) or an (i, j)-edge, if $\deg(u) \leq i$ and $\deg(v) \leq j$. A path on three vertices u, v, and w is a path of type (i, j, k) or an (i, j, k)-path if $\deg(u) \leq i$, $\deg(v) \leq j$, and $\deg(w) \leq k$. Let $\delta(G) = \delta$ be the minimum vertex degree in G, and $w_k(G)$ or simply w_k be the minimum sum of degrees of vertices of a path on k vertices. The girth $g(G)$ or g of G is the length of a shortest cycle in G.

Let $(v_1, v_2, \ldots, v_k)$ be a facial walk of a k-face α. A *facial degree sequence* of α is a sequence of non-negative integers $(a_1, a_2, \ldots, a_k)$ such that $\deg(v_i) = a_i$ for all $i = 1, \ldots, k$.

Already in 1904, Wernicke [12] proved that every normal plane map G such that $\delta(G) = 5$ contains a 5-vertex adjacent to a 6^--vertex, and Franklin [6] strengthened this to the existence of a $(6, 5, 6)$-path in such normal plane maps. As concerns the existence of (a, b)-paths with bounded a and b (i.e. light edges) in a normal plane map, the effort of Lebesgue [11], Kotzig [10], Barnette [7] has flowed in the following theorem by Borodin [3]

Theorem 1. *Every normal plane map contains an edge of one of the following types:* $(3, 10)$, $(4, 7)$, *or* $(5, 6)$. *The bounds* 10, 7, *and* 6 *are tight.*

For the more detailed history of the research in this direction see recent papers [4] or [9].

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The graphs $K_{1,r}$ and $K_{2,r}$, $r \geq 2$, have only $(1, r)$-edges and $(2, r)$-edges, respectively. These examples show that the requirement on minimum degree cannot be omitted when trying to extend Theorem 1 to wider families of plane graphs with minimum degree at least two. The situation changes significantly if we involve the girth of plane graphs into the game. Namely we have

Theorem 2. *Every connected plane graph of minimum degree $\delta(G) \geq 2$ and girth $g(G) \geq g$ contains an edge of the type*

(i) $(2, 5)$ *or* $(3, 3)$, *if* $g = 5$,
(ii) $(2, 5)$, *if* $g = 6$,
(iii) $(2, 3)$, *if* $g \in \{7, 8, 9, 10\}$,
(iv) $(2, 2)$, *if* $g \geq 11$.

Moreover, no parameter in this description can be lowered.

In this paper we are interested also in 3-paths. The motivation for our research has come from the following three results

Theorem 3 (*Ando, Iwasaki, Kaneko [1]*)**.** *Every 3-polytope satisfies $w_3 \leq 21$, which is tight.*

Theorem 4 (*Jendrol' [8]*)**.** *Every 3-polytope has a 3-path of one of the following types:* $(10, 3, 10)$, $(7, 4, 7)$, $(6, 5, 6)$, $(3, 4, 15)$, $(3, 6, 11)$, $(3, 8, 5)$, $(3, 10, 3)$, $(4, 4, 11)$, $(4, 5, 7)$, *or* $(4, 7, 5)$.

Theorem 5 (*Borodin et al. [4]*)**.** *Every normal plane map without two adjacent 3-vertices lying in two common 3-faces has a 3-path of one of the following types:* $(3, 4, 11)$, $(3, 7, 5)$, $(3, 10, 4)$, $(3, 15, 3)$, $(4, 4, 9)$, $(6, 4, 8)$, $(7, 4, 7)$, *or* $(6, 5, 6)$. *Moreover, no parameter of this description can be improved.*

Theorems 3–5 deal with graphs having minimum degree at least three. We drop this requirement and consider simple plane graphs with minimum degree at least two. The purpose of this paper is to describe 3-paths in connected plane graphs G with $\delta(G) \geq 2$ and $g(G) \geq 5$. Our next main result is the following

Theorem 6. *Every connected plane graph of minimum degree $\delta(G) \geq 2$ and girth $g(G) \geq g$ has a 3-path of one of the following types*

(i) $(2, \infty, 2)$, $(2, 2, 6)$, $(2, 3, 5)$, $(2, 4, 4)$, *or* $(3, 3, 3)$, *if* $g = 5$,
(ii) $(2, 2, \infty)$, $(2, 3, 5)$, $(2, 4, 3)$, *or* $(2, 5, 2)$, *if* $g = 6$,
(iii) $(2, 2, 6)$, $(2, 3, 3)$, *or* $(2, 4, 2)$, *if* $g = 7$,
(iv) $(2, 2, 5)$ *or* $(2, 3, 3)$, *if* $g \in \{8, 9\}$,
(v) $(2, 2, 3)$ *or* $(2, 3, 2)$, *if* $g \geq 10$, *and*
(vi) $(2, 2, 2)$, *if* $g \geq 16$.

The rest of the paper is organized as follows: In Section 2 we give a proof of Theorem 2. In remaining sections the proof of Theorem 6 is performed according to the girth of the graph. For $g = 5$ it is given in Section 3, for $g = 6$ in Section 4, for $g = 7$, 8 and 9 in Section 5, for $g \geq 10$ in Section 6 and for $g \geq 16$ in Section 7. At the end of each section the quality of the corresponding types of paths is discussed.

2. Proof of Theorem 2

We prove a little more than Theorem 2(i) states.

Theorem 7. *Let G be a connected plane pseudograph G with $\delta(G) \geq 2$ and minimum face size at least five. Then G contains an (a, b)-edge of type $(2, 5)$ or type $(3, 3)$. Moreover, none of the parameters in these pairs can be lowered.*

Proof. Suppose it is not true. Then there exists a plane pseudograph $G = (V, E, F)$ satisfying the condition of the theorem but having no edge of the mentioned types. We will use the Discharging method (see e.g. [5]).

$$\sum_{u \in V(G)} (2\deg(u) - 6) + \sum_{\alpha \in F(G)} (\deg(\alpha) - 6) = -12 \tag{2.1}$$

Using the consequence of the Euler formula (2.1) we put the initial charge $ch(u) = 2\deg(u) - 6$ and $ch(\alpha) = \deg(\alpha) - 6$ to the vertices v and faces α of G, respectively. The sum of these charges is negative. These initial charges are locally redistributed according to the following rule R.

R: Every k-vertex v gives $\frac{2k-6}{k}$ to each incident face α. If v appears on the facial walk of α t times, then v gives this charge t times to α.

Next we show that the new charge $\varphi(x)$ of any element $x \in V(G) \cup F(G)$ is non-negative, which will be a contradiction.

It is easy to see that, after applying the rule R, $\varphi(v) = 0$ for every $v \in V(G)$.

Because every edge of G contains a 4^+-vertex, any k-face α satisfies

$$\varphi(\alpha) \geq k - 6 + \left\lceil \frac{k}{2} \right\rceil \times \frac{1}{2} - \left\lfloor \frac{k}{2} \right\rfloor \times 1.$$