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Implicit representations and factorial properties of graphs

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ABSTRACT

The idea of implicit representation of graphs was introduced in Kannan et al. (1992) and can be defined as follows. A representation of an n-vertex graph G is said to be implicit if it assigns to each vertex of G a binary code of length $O(\log n)$ so that the adjacency of two vertices is a function of their codes. Since an implicit representation of an n-vertex graph uses $O(n \log n)$ bits, any class of graphs admitting such a representation contains $2^{O(n \log n)}$ labelled graphs with n vertices. In the terminology of Balogh et al. (2000) such classes have at most factorial speed of growth. In this terminology, the implicit graph conjecture can be stated as follows: every class with at most factorial speed of growth which is hereditary admits an implicit representation. The question of deciding whether a given hereditary class has at most factorial speed of growth is far from being trivial. In the present paper, we introduce a number of tools simplifying this question. Some of them can be used to obtain a stronger conclusion on the existence of an implicit representation. We apply our tools to reveal new hereditary classes with the factorial speed of growth. For many of them we show the existence of an implicit representation.

1. Introduction

We study simple graphs, i.e. undirected graphs without loops and multiple edges. We denote by $M = M_G$ the adjacency matrix of a graph G and by $m(u, v) = m_G(u, v)$ the element of M corresponding to vertices u and v, i.e. $m(u, v) = 1$ if u and v are adjacent and $m(u, v) = 0$ otherwise.

Every simple graph on n vertices can be represented by a binary word of length $\binom{n}{2}$ (half of the adjacency matrix), and if no a priory information about the graph is known, this representation is best possible in terms of its length. However, if we know that our graph belongs to a particular class (possesses a particular property), this representation can be shortened. For instance, the Prüfer code allows representing a labelled tree with n vertices by a word of length $(n - 2) \log n$ (in binary encoding).[1] For labelled graphs, i.e. graphs with vertex set $\{1, 2, \ldots, n\}$, we need $\log n$ bits for each vertex just to represent its label. That is why representations of n-vertex graphs from a specific class requiring $O(\log n)$ bits per vertex have been called in [6] *implicit*.

Throughout the paper by representing a graph we mean its coding, i.e. representing by a word in a finite alphabet (in our case the alphabet is always binary). Moreover, we assume that different graphs are mapped to different words (i.e. the mapping is injective) and that the graph can be restored from its code. For an implicit representation, we additionally require

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1 All logarithms in this paper are of base 2.

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that the code of the graph consists of the codes of its vertices, each of length $O(\log n)$, and that the adjacency of two vertices, i.e. the element of the adjacency matrix corresponding to these vertices, can be computed from their codes.

Not every class of graphs admits an implicit representation, since a bound on the total length of the code implies a bound on the number of graphs admitting such a representation. More precisely, only classes containing $2^{O(n\log n)}$ graphs with n vertices can admit an implicit representation. However, this restriction does not guarantee that graphs in such classes can be represented implicitly. A simple counter-example can be found in [13]. Even with further restriction to *hereditary classes*, i.e. those that are closed under taking induced subgraphs, the question is still not so easy. The authors of [6], who introduced the notion of an implicit representation, conjectured that every hereditary class with $2^{O(n\log n)}$ graphs on n vertices admits an implicit representation, and this conjecture is still open.

In the terminology of [4], hereditary classes containing $2^{O(n\log n)}$ labelled graphs on n vertices are at most *factorial*, i.e. have at most factorial speed of growth. Classes with speeds lower than factorial are well studied and have a very simple structure. The family of factorial classes is substantially richer and the structure of classes in this family is more diverse. It contains many classes of theoretical or practical importance, such as line graphs [8], interval graphs [13], permutation graphs [13], threshold graphs, forests, planar graphs and, even more generally, all proper minor-closed graph classes [11], all classes of graphs of bounded vertex degree, of bounded clique-width [3], etc.

In spite of the crucial importance of the family of factorial classes, except the definition very little can be said about this family in general, and the membership in this family is an open question for many particular graph classes, for instance, for P_7-free bipartite graphs. To simplify the study of this question, in the present paper we introduce a number of tools and apply them to reveal new members of this family. For some of them, we do even better and find an implicit representation.

The organization of the paper is as follows. In the rest of this section, we introduce basic definitions and notations related to the topic of the paper. In Section 2, we define our tools and then in Section 3 we apply them to discover new factorial classes of graphs and new classes admitting an implicit representation.

The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Given a vertex $v \in V(G)$, we denote by $N(v)$ the neighbourhood of v, i.e. the set of vertices adjacent to v. For a subset $S \subset V(G)$, we denote by $N(S)$ the neighbourhood of S, i.e. the set of vertices outside of S that have at least one neighbour in S. The degree of v is the number of its neighbours, i.e. $|N(v)|$, and co-degree is the number of its non-neighbours, i.e. its degree in the complement of the graph. As usual, we denote by C_n, P_n, K_n the chordless cycle, the chordless path and the complete graph with n vertices, respectively. Also, O_n stands for the complement of K_n, i.e. the empty (edgeless) graph with n vertices, and $S_{i,j,k}$ for the tree with three vertices of degree 1 being of distance i, j, k from the only vertex of degree 3.

In a graph, a *clique* is a set of pairwise adjacent vertices and an *independent set* is a set of vertices no two of which are adjacent. A graph G is a *split* graph if $V(G)$ can be partitioned into a clique and an independent set, and G is *bipartite* if $V(G)$ can be partitioned into at most two independent sets. A complete bipartite graph with parts (independent sets) of size n and m is denoted by $K_{n,m}$. We refer to the complement of a bipartite graph as *co-bipartite*.

We say that a graph H is an induced subgraph of a graph G if $V(H) \subseteq V(G)$ and two vertices of H are adjacent if and only if they are adjacent in G. If G contains no induced subgraph isomorphic to H, we say that G is H-free. Given a set M of graphs, we denote by $Free(M)$ the class of graphs containing no induced subgraphs isomorphic to graphs in the set M. Clearly, for any set M, the class $Free(M)$ is *hereditary*, i.e. closed under taking induced subgraphs. The converse is also true: for any hereditary class X there is a set M such that $X = Free(M)$. Moreover, the minimal set M with this property is unique. We call M the set of *forbidden induced subgraphs* for the class X.

Given a class X, we write X_n for the number of labelled graphs in X and call X_n the *speed* of X. The speed of hereditary classes (also known as hereditary properties)[2] has been extensively studied in the literature. In particular, paper [12] shows that the rates of the speed growth constitute discrete layers and distinguishes the first four of these layers: constant, polynomial, exponential and factorial. Independently, similar results have been obtained by Alekseev in [1]. Moreover, Alekseev provided the first four layers with the description of all minimal classes, i.e. he identified in each layer a family of classes every hereditary subclass of which belongs to a lower layer (see also [4] for some more involved results). In particular, the factorial layer has 9 minimal classes, three of which are subclasses of bipartite graphs, three others are subclasses of co-bipartite graphs (complements of bipartite graphs) and the remaining three are subclasses of split graphs. The three minimal factorial classes of bipartite graphs are:

- $P^1 = Free(K_3, K_{1,2})$, the class of graphs of vertex degree at most 1,
- P^2, the class of "bipartite complements" of graphs in P^1, i.e. the class of bipartite graphs in which every vertex has at most one non-neighbour in the opposite part,
- $P^3 = Free(C_3, C_5, 2K_2)$, the class of $2K_2$-free bipartite graphs, also known as chain graphs for the property that the neighbourhoods of vertices in each part form a chain.

The structure of graphs in these classes and in the related subclasses of split and co-bipartite graphs is very simple and hence the problem of deciding whether a hereditary class has at least factorial speed of growth admits an easy solution. In the next section, we introduce a number of tools that can be helpful in deciding whether the speed of a hereditary class is at most factorial.

[2] Throughout the paper, we use the two terms, hereditary classes and hereditary properties, interchangeably.