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Forests and trees among Gallai graphs

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ABSTRACT

The Gallai graph $\mathrm{Gal}(G)$ of a graph G has the edges of G as its vertices, and two distinct vertices e and f of $\mathrm{Gal}(G)$ are adjacent in $\mathrm{Gal}(G)$ if the edges e and f of G are adjacent in G but do not span a triangle in G. In the present paper we characterize those graphs whose Gallai graphs are forests or trees, respectively.

1. Introduction

We consider finite, simple, and undirected graphs and use standard terminology and notation as in [15]. For a graph G, the *Gallai graph* $\mathrm{Gal}(G)$ of G has the edges of G as its vertices, that is, $V(\mathrm{Gal}(G)) = E(G)$, and two distinct vertices e and f of $\mathrm{Gal}(G)$ are adjacent in $\mathrm{Gal}(G)$ if the edges e and f of G are adjacent in G but do not span a triangle in G. Gallai graphs were introduced by Gallai [6] in connection with cocomparability graphs and were used by Chvátal and Sbihi [5] in their polynomial-time recognition algorithm for claw-free perfect graphs. Obviously, the Gallai graph $\mathrm{Gal}(G)$ is a spanning subgraph of the well-known *line graph* $L(G)$ of G, which is the intersection graph of the set of pairs of vertices forming edges in G. The *anti-Gallai graph* or *triangular line graph* of G is the complement of $\mathrm{Gal}(G)$ in $L(G)$, that is, it has vertex set $E(G)$ and edge set $E(L(G)) \setminus E(\mathrm{Gal}(G))$. Anti-Gallai graphs were introduced by Jarret [8].

Gallai and anti-Gallai graphs were studied in [2,9–11]. The graphs whose Gallai graphs are bipartite were characterized in [13]. In the present paper we characterize those graphs whose Gallai graphs are forests or trees, respectively.

Our main results are as follows.

Theorem 1. *The Gallai graph* $\mathrm{Gal}(G)$ *of a graph* G *is a forest if and only if* G *is an* $\{F_1, \ldots, F_9\}$*-free chordal graph (see Fig. 1).*

A vertex of degree 1 is an *end-vertex*. The *gem* is the graph that arises by removing the two end-vertices from F_7. A set U of vertices of a graph G is *homogeneous* if every vertex in $V(G) \setminus U$ is adjacent either to all vertices in U or to no vertex in U. A homogeneous set U is *nontrivial* if $|U| \notin \{0, 1, |V(G)|\}$.

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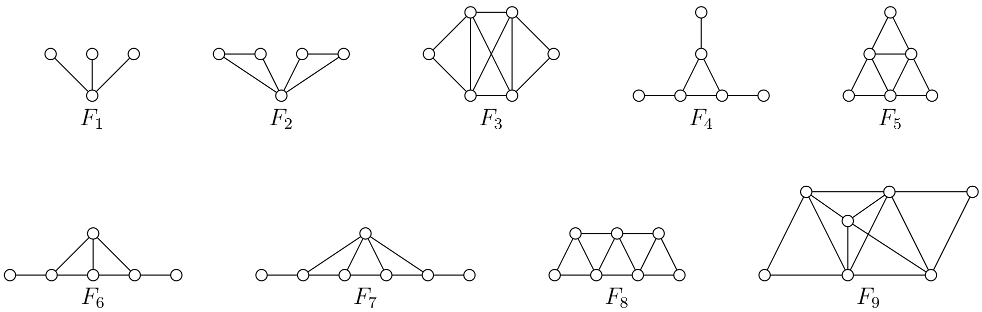

Fig. 1. Forbidden induced subgraphs.

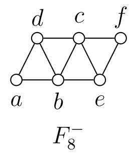

Fig. 2. The graph F_8^-.

Theorem 2. *For a graph G without isolated vertices, the following statements are equivalent:*

(i) *The Gallai graph* $\mathrm{Gal}(G)$ *of* G *is a tree.*
(ii) *Every nontrivial homogeneous set in G is independent, and G is an* $\{F_1, \ldots, F_9\}$*-free chordal graph.*
(iii) *Either G is the graph* F_8^- *in Fig. 2, or G is connected and satisfies the following conditions:*
 – *Every block of G is isomorphic to* K_2, K_3, *or a gem.*
 – *Every cut-vertex of G lies in at most two blocks and has degree at most 3 in G.*
 – *Every block of G that is isomorphic to* K_3 *contains exactly two cut-vertices of G.*
 – *Every block of G that is isomorphic to a gem contains exactly one cut-vertex of G.*

The rest of the paper is devoted to the proofs of the above results.

2. Preliminaries

Before we proceed to the proofs of our results, we make two immediate observations.

Observation 3. *Every graph is an induced subgraph of some Gallai graph.*

In fact, if H is a graph and the graph G has vertex set $V(H) \cup \{x\}$ such that all vertices in $V(H)$ are neighbors of x in G and $G-x$ is the complement of H, then the subgraph of $\mathrm{Gal}(G)$ induced by the edges of G that are incident with x is isomorphic to H (indeed, $\mathrm{Gal}(G)$ is the disjoint union of H and $\mathrm{Gal}(\overline{H})$). This observation explains to some extent why the characterization of Gallai graphs is difficult. In particular, it means that there are no forbidden induced subgraphs for Gallai graphs.

Observation 4. *If* G' *is an induced subgraph of a graph G, then* $\mathrm{Gal}(G')$ *is an induced subgraph of* $\mathrm{Gal}(G)$.

This follows immediately from the definition.

We include a proof of a known but somewhat inaccessible result. It requires a more easily accessible (but more difficult) lemma. For a component Q of $\mathrm{Gal}(G)$, let $V_Q(G)$ denote the set of vertices in G that are incident with an edge of G that is a vertex of Q.

Lemma 5 (*Arditti and Jung, Lemma 4 in [3]*)**.** *If C and D are distinct components of* $\mathrm{Gal}(G)$, *and the edge xy of G is a vertex of C that satisfies* $x, y \in V_D(G)$, *then* $V_C(G)$ *is a proper subset of* $V_D(G)$.

Lemma 6 (*Le [9]*)**.** *If G is a graph without isolated vertices, then* $\mathrm{Gal}(G)$ *is connected if and only if every nontrivial homogeneous set in G is independent.*

Proof. Let G be a graph without isolated vertices.

Necessity. By the definition of $\mathrm{Gal}(G)$, a vertex of $\mathrm{Gal}(G)$ corresponding to an edge induced by a homogeneous set U of G cannot be adjacent to a vertex corresponding to an edge not induced by U. If U is a nontrivial homogeneous set and G has no isolated vertex, then G has at least one edge not induced by U. Hence, if U is not an independent set in G, then $\mathrm{Gal}(G)$ is not connected.