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On coloring box graphs

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ABSTRACT

We consider the chromatic number of a family of graphs we call box graphs, which arise from a box complex in n-space. It is straightforward to show that any box graph in the plane has an admissible coloring with three colors, and that any box graph in n-space has an admissible coloring with $n + 1$ colors. We show that for box graphs in n-space, if the lengths of the boxes in the corresponding box complex take on no more than two values from the set $\{1, 2, 3\}$, then the box graph is 3-colorable, and for some graphs three colors are required. We also show that box graphs in 3-space which do not have cycles of length four (which we call "string complexes") are 3-colorable.

1. Introduction and results

There are many geometrically-defined graphs whose chromatic numbers have been studied. Perhaps the most famous such example is the Four Color Theorem, which states that any planar graph is 4-colorable [1]. Another famous example is the chromatic number of the plane. More specifically, a graph $G = (V, E)$ is defined where $V = \mathbb{R}^2$ and $(x, y) \in E$ precisely when $\|x - y\|_2 = 1$ (where $\|\cdot\|_2$ is the usual Euclidean norm in the plane). Through simple geometric constructions, one can show that $4 \leq \chi(G) \leq 7$ for this graph, although the precise value is still not known; see [8], for example.

In this article, we consider graphs that arise from box complexes. We first define what a box complex is:

Definition 1. An n-dimensional *box* is a set $B \subset \mathbb{R}^n$ that can be defined as:

$$B = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}$$

where $a_i < b_i$ for $i = 1, 2, \ldots, n$.

An n-dimensional *box complex* is a set of finitely many n-dimensional boxes $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ such that if the intersection of two boxes $B_i \cap B_j$ is nonempty, then $B_i \cap B_j$ is a face (of any dimension) of both B_i and B_j, for any i and j (see Fig. 1).

Now we can define a box graph:

Definition 2. An n-dimensional *box graph* is a graph defined on an n-dimensional box complex. The box graph $G(\mathcal{B}) = (V, E)$ defined on the box complex $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ is the undirected graph whose vertex set is the boxes:

$$V = \{B_1, B_2, \ldots, B_m\}$$

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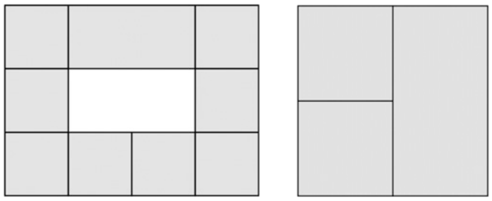
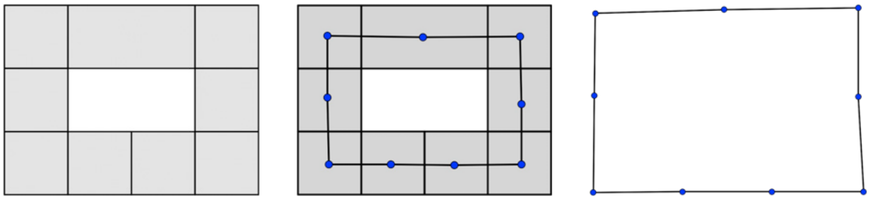


(a) A box complex. (b) Not a box complex.

Fig. 1. Examples in $\mathbb{R}^2$.



(a) A box complex. (b) The box graph inside the box complex. (c) Just the box graph.

Fig. 2. Defining a 2-dimensional box graph.

and whose edges $(B_i, B_j) \in E$ record when $B_i \cap B_j$ is an $(n-1)$-dimensional face of both B_i and B_j. In other words, the box graph is the dual graph of the box complex, and the colorings we are considering are in some sense "solid colorings."

When it eases understanding, we may use the terms box complex and box graph interchangeably. We also may use boxes and vertices interchangeably.

The following proposition shows that, as far as the corresponding box graphs are concerned, we may as well restrict ourselves to box complexes where each of the vertices of the boxes has integer coordinates (and thus all boxes have integer lengths).

Proposition 1. *Let $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ be a box complex and let $G(\mathcal{B}) = (V, E)$ be its corresponding box graph. There exists a box complex $\{C_1, C_2, \ldots, C_m\}$ where the vertices of each C_i $(i = 1, 2, \ldots, m)$ have all integer coordinates, such that the box graph corresponding to complex $\{C_1, C_2, \ldots, C_m\}$ is the same graph G.*

We will prove Proposition 1 in Section 2.

We ask the following natural question:

Question 1. What is the minimum number of colors k that are required so that every n-dimensional box graph has an admissible k-coloring?

From Fig. 2(c), we can see that three colors may be necessary to color a 2-dimensional box graph. In fact, as we will prove in Section 2, three colors are also sufficient:

Proposition 2. *Any box graph in n-space has an admissible coloring with $n + 1$ colors.*

Our goal is to answer Question 1 in dimension 3, which is still open. In the case where the "boxes" are zonotopes (as opposed to right-angled bricks), sometimes 4 colors are needed [4], and in the case where the "boxes" are now touching spheres, the chromatic number is between 5 and 13 [2]. Analogously, for simplicial complexes in $\mathbb{R}^n$, $n+1$ colors suffice [6]. We suspect that any 3-dimensional box graph is 3-colorable, and we can show that this is true for a few families of 3-dimensional box graphs. The following are the main results of this paper:

Theorem 1. *Let G be an n-dimensional box graph such that the lengths of all of the boxes in the corresponding box complex take on no more than two values from the set $\{1, 2, 3\}$. That is, all the side lengths of the boxes are 1 or 2, or all the side lengths are 1 or 3, or all the side lengths are 2 or 3. Then G is 3-colorable.*

Theorem 2. *Let G be a 3-dimensional box graph that has no cycles on four vertices. Then G is 3-colorable.*

The rest of this paper is organized as follows: in Section 2 we will state and prove some straightforward results on box graphs. We will prove Theorem 1 in Section 3, and we will prove Theorem 2 in Section 4.

2. Straightforward results on box graphs

As promised, we will start with proofs of Propositions 1 and 2.