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Tournaments associated with multigraphs and a theorem of Hakimi

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ABSTRACT

A tournament of order n is usually considered as an orientation of the complete graph K_n. In this note, we consider a more general definition of a tournament that we call a C-tournament, where C is the adjacency matrix of a multigraph G, and a C-tournament is an orientation of G. The score vector of a C-tournament is the vector of outdegrees of its vertices. In 1965 Hakimi obtained necessary and sufficient conditions for the existence of a C-tournament with a prescribed score vector R and gave an algorithm to construct such a C-tournament which required, however, some backtracking. We give a simpler and more transparent proof of Hakimi's theorem, and then provide a direct construction of such a C-tournament which works even for weighted graphs.

1. Introduction

Let K_n be the complete graph with vertices $\{1, 2, \ldots, n\}$. A *tournament* of order n is an orientation of K_n. Its adjacency matrix, a *tournament matrix*, is an $n \times n$ (0, 1)-matrix $T = [t_{ij}]$ such that $T + T^t = J_n - I_n$ where J_n is the $n \times n$ matrix of all 1s. We shall not distinguish between a tournament and a tournament matrix and usually refer to both as tournaments and label both as T. The adjacency matrix presupposes a listing of the vertices in a specified order; changing the order of the vertices replaces T with PTP^t for some permutation matrix P. The *score vector* of T is $R = (r_1, r_2, \ldots, r_n)$ where r_i is the number of 1s in row i, that is, the ith row sum. The score vector of T is the vector of outdegrees of the vertices of T; the outdegrees determine the indegrees, since the sum of the outdegree and indegree of a vertex is $n-1$. One of the best known theorems for tournaments is Landau's theorem [15] of 1953 which asserts that a vector $R = (r_1, r_2, \ldots, r_n)$ of nonnegative integers is the score vector of a tournament of order n if and only if

$$\sum_{i\in J} r_i \geq \binom{|J|}{2} \ (J \subseteq \{1, 2, \ldots, n\}), \quad \text{with equality if } J = \{1, 2, \ldots, n\}. \tag{1}$$

If we assume that $r_1 \leq r_2 \leq \cdots \leq r_n$, which we can get by reordering, then (1) is equivalent to

$$\sum_{i=1}^{k} r_i \geq \binom{k}{2} \ (k = 1, 2, \ldots, n), \quad \text{with equality if } k = n. \tag{2}$$

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A generalization of Landau's theorem to 2-tournaments is given by Avery [1] (see also [3, pp. 267–274]). Let $2G$ denote the multigraph obtained from a graph G by doubling each edge, that is, each edge between a pair of vertices becomes two edges. Define a 2-tournament to be an orientation of $2K_n$. Then Avery proved that a vector $R = (r_1, r_2, \ldots, r_n)$ of nonnegative integers with $r_1 \le r_2 \le \cdots \le r_n$ is the score vector of a 2-tournament if and only if

$$\sum_{i=1}^{k} r_i \ge 2\binom{k}{2} \ (k = 1, 2, \ldots, n) \quad \text{with equality if } k = n. \tag{3}$$

In fact, others [17] have considered the generalizations of Landau's theorem to *p-tournaments*, that is, to orientations of pK_n for an integer $p \ge 2$, with the various proofs for Landau's theorem ($p = 1$) carrying over without much change (see [18] for a survey of proofs; also see [8]). In the case of $p = 2$, Avery proves more about the existence of a 2-tournament with score vector R and indeed gives an algorithm to construct an orientation of $2K_n$ with the smallest number of 2s possible in its matrix (also see its exposition in [3, pp. 267–274]). Iványi [11,12,14], and Iványi and Schoenfield [13], study score sequences which arise from orientations of graphs whose degrees are in a prescribed interval $[p, q]$ where p and q are integers with $p \le q$. Thus, when $p = q$ these are score sequences of p-tournaments. In [10], it is shown how the existence theorem for p-tournaments with $p \ge 2$ follows from the existence theorem for 1-tournaments.

It does not seem to be well-known, at least judging from references in papers discussing scores in tournaments, that in 1965 Hakimi [9] proved an even more general theorem. (Hakimi does not reference Landau's theorem so, apparently, it was unknown to him.) Hakimi considered an arbitrary multigraph G of order n in which multiple edges but not loops are allowed and the score vector of an orientation $\vec{G}$ of G, that is, its vector of outdegrees. (The indegree of a vertex in $\vec{G}$ is determined by its outdegree since their sum is its degree in G.) Let the vertices of G be $\{1, 2, \ldots, n\}$. Hakimi's theorem asserts that a vector $R = (r_1, r_2, \ldots, r_n)$ of nonnegative integers is the score vector of an orientation of G if and only if

$$r(J) \ge E(J) \ (J \subseteq \{1, 2, \ldots, n\}) \quad \text{with equality if } J = \{1, 2, \ldots, n\}, \tag{4}$$

where $r(J) = \sum_{i \in J} r_i$ and $E(J)$ is the number of edges in the subgraph $G(J)$ of G induced by the vertices in J. If $G = K_n$, then $E(J) = \binom{|J|}{2}$, and thus Hakimi's theorem reduces to Landau's theorem. One of the few papers citing [9] is the paper [7] by Entringer and Tolman where a brief survey of orientations of graphs is presented, and a theorem concerning orientations of graphs with indegrees and outdegrees of vertices in prescribed intervals is proved.

Let $C = [c_{ij}]$ be an integral nonnegative symmetric matrix with 0s on the main diagonal where C is regarded as the adjacency matrix of a multigraph G with vertex set $\{1, 2, \ldots, n\}$ in which vertex i is joined to vertex j by c_{ij} edges for each $i \ne j$. Note that it is possible that for some $i \ne j$, $c_{ij} = 0$ so that there are no edges between vertices i and j. Cruse [5] defined a *C-tournament* to be an orientation of G; if, as above, we do not distinguish between an oriented graph and its adjacency matrix, a C-tournament is an $n \times n$ integral nonnegative matrix T such that $T + T^t = C$. In a C-tournament $T = [t_{ij}]$, players i and j play a prescribed number c_{ij} of games, and player i wins t_{ij} of these games and player j wins the other $c_{ij} - t_{ij}$ games. Thus if we take $C = p(J_n - I_n)$ for some positive integer p, we get p-tournaments. The score vector of a C-tournament is $R = (r_1, r_2, \ldots, r_n)$ where r_i is the number of games won by player i, that is, the sum of the entries of T in row i. Using linear programming techniques, Cruse [5] characterized score vectors of C-tournaments as follows: A vector $R = (r_1, r_2, \ldots, r_n)$ of nonnegative integers is the score vector of a C-tournament if and only if

$$\sum_{i \in J} r_i \ge \sum_{i,j \in J, i<j} c_{ij} \ \ (J \subseteq \{1, 2, \ldots, n\}) \quad \text{with equality if } J = \{1, 2, \ldots, n\}.$$

This is equivalent to Hakimi's theorem. Based on the approach in [5], Cruse [6] provides a polynomial algorithm for a C-tournament with score vector R.

In the next section we give a proof of the theorem of Hakimi (using the terminology of Cruse) along the lines of the proofs of Landau's theorem given by Mahmoodian [16] and Thomassen [19]. We also sketch a proof using Rado's theorem on independent transversals of a family of subsets of a matroid, along the lines of the proof of Landau's theorem given in [4]. In the final section we give and illustrate a method to construct a C-tournament with a prescribed score vector when such a C-tournament exists. In fact, this construction works assuming only that the entries of C and R are nonnegative real numbers. This shows that (4) is also necessary and sufficient for the existence of a generalized C-tournament with score vector (row sum vector) R, and extends the existence result for generalized tournaments in [2,17].

2. Existence of *C*-tournaments

We now formally state and prove the theorem for the existence of a C-tournament with a prescribed score vector $R = (r_1, r_2, \ldots, r_n)$. Recall that for $J \subseteq \{1, 2, \ldots, n\}$, $r(J) = \sum_{i \in J} r_i$ and, where $C = [c_{ij}]$, we also define $c(J) = \sum_{i,j \in J, i<j} c_{ij}$.

Theorem 1. *Let $C = [c_{ij}]$ be an $n \times n$ integral nonnegative symmetric matrix with 0s on the main diagonal. A vector $R = (r_1, r_2, \ldots, r_n)$ of nonnegative integers is the score vector of a C-tournament if and only if*

$$r(J) \ge c(J) \ (J \subseteq \{1, 2, \ldots, n\}) \quad \textit{with equality if } J = \{1, 2, \ldots, n\}. \tag{5}$$