Discrete Mathematics 338 (2015) 274–290

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Cycle-maximal triangle-free graphs

Stephane Durocher [a], David S. Gunderson [b], Pak Ching Li [a], Matthew Skala [a,*]

[a] Department of Computer Science, E2-445 EITC, University of Manitoba, Winnipeg, Manitoba, Canada, R3T 2N2

[b] Department of Mathematics, 342 Machray Hall, 186 Dysart Road, University of Manitoba, Winnipeg, Manitoba, Canada, R3T 2N2

ARTICLE INFO

Article history:
Received 18 October 2013
Received in revised form 15 August 2014
Accepted 4 October 2014
Available online 28 October 2014

Keywords:
Extremal graph theory
Cycle
Triangle-free
Regular graph
Matrix permanent
#*P*-complete

ABSTRACT

We conjecture that the balanced complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ contains more cycles than any other n-vertex triangle-free graph, and we make some progress toward proving this. We give equivalent conditions for cycle-maximal triangle-free graphs; show bounds on the numbers of cycles in graphs depending on numbers of vertices and edges, girth, and homomorphisms to small fixed graphs; and use the bounds to show that among regular graphs, the conjecture holds. We also consider graphs that are close to being regular, with the minimum and maximum degrees differing by at most a positive integer k. For $k = 1$, we show that any such counterexamples have $n \leq 91$ and are not homomorphic to C_5; and for any fixed k there exists a finite upper bound on the number of vertices in a counterexample. Finally, we describe an algorithm for efficiently computing the matrix permanent (a #*P*-complete problem in general) in a special case used by our bounds.

1. Introduction

Many algorithmic problems that are computationally difficult on graphs can be solved easily in polynomial time when the graph is acyclic. Limiting input to trees (connected acyclic graphs) or forests (acyclic graphs), however, is often too restrictive; many of these problems remain efficiently solvable when the graph is "nearly" a tree [6–8,21]. Various notions exist formalizing how close a given graph is to being a tree, including bounded treewidth (partial k-trees), k-connectivity, and number of cycles.

The problem of evaluating $c(G)$ for a given graph is #*P*-complete, equivalent in difficulty to counting the certificates of an *NP*-complete decision problem, even though the problem of testing for the existence of a single cycle is trivially polynomial-time. Existence of a cycle is a graph property definable in monadic second-order logic. By the result known as Courcelle's Theorem [15], such properties can be decided in linear time for graphs of bounded treewidth, and as described by Arnborg, Lagergren, and Seese, the counting versions are also linear-time for fixed treewidth [6]. On the other hand, if we parameterize by length of the cycles instead of structure of the graph, Flum and Grohe [19] give evidence against fixed-parameter tractability: they show that counting cycles of length k is #$W[1]$-complete, with no $(f(k) \cdot n^c)$-time algorithm unless the Exponential Time Hypothesis fails.

When no restrictions are imposed on the graph, the number of cycles in an n-vertex graph is maximized by the complete graph on n vertices, K_n. In this case the number of cycles is easily seen to be

$$\sum_{i=3}^{n}\left(\binom{n}{i}\frac{(i-1)!}{2}\right) = n!\sum_{i=3}^{n}\frac{1}{2i(n-i)!}. \tag{1}$$

* Corresponding author.
E-mail addresses: durocher@cs.umanitoba.ca (S. Durocher), gunderso@cc.umanitoba.ca (D.S. Gunderson), lipakc@cs.umanitoba.ca (P.C. Li), mskala@ansuz.sooke.bc.ca (M. Skala).

http://dx.doi.org/10.1016/j.disc.2014.10.002
0012-365X/

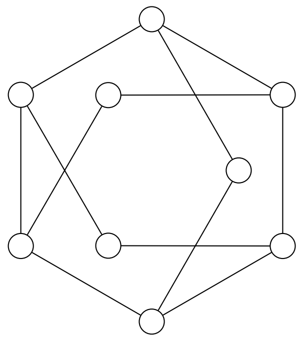

Fig. 1. The Petersen graph minus one vertex, which contains a C_6 that cannot be bridged without creating a triangle.

The bound (1) can be refined by introducing additional parameters. Previous results include bounds on the number of cycles in terms of n, m, δ, and Δ (the number of vertices, number of edges, minimum degree, and maximum degree of G, respectively) [16,20,36], as well as bounds on the number of cycles for various classes of graphs, including k-connected graphs [25], Hamiltonian graphs [29,33], planar graphs [1,2], series–parallel graphs [27], and random graphs [34].

A graph's cycles can be classified by length. For each value of i, the summand in (1) corresponds to the number of cycles of length i in K_n. If short cycles are disallowed, the number of long cycles possible is also reduced. Every graph G of girth g that contains two or more cycles has $n \geq 3g/2 - 1$ vertices or, equivalently, if $g > 2(n + 1)/3$, then G has at most one cycle [9]. The bound on the number of cycles increases as g decreases. As mentioned earlier, the case $g = 3$ is maximized by K_n for which the number of cycles is exactly (1). Can the maximum number of cycles be expressed exactly or bounded tightly as a function of arbitrary values for n and g? Even when $g = 4$ the maximum number of possible cycles is unknown. Graphs of girth four or greater are exactly the triangle-free graphs. One goal of this research program is to show that the number of cycles in a triangle-free n-vertex graph is maximized by the complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$, and the results in this paper represent significant progress toward that goal.

We first encountered the problem of bounding the number of cycles as a function of n and g when examining path-finding algorithms on graphs. A tree traversal can be achieved by applying a right-hand rule (e.g., after reaching a vertex v via its ith edge, depart along its $(i+1)$st edge). Traversing a graph using only local information at each vertex is significantly more difficult in graphs with cycles. A successful traversal can be guaranteed, however, if the local neighbourhood of every vertex v is tree-like within some distance k from v (e.g., the graph has girth $g \geq 2k + 1$) and that a fixed upper bound is known on the number of possible cycles along paths that join pairs of leaves outside each such local tree (Bose, Carmi, and Durocher [9] give a more formal discussion). Deriving a useful bound on this number of cycles led to the work presented in this paper.

In any graph, every chordless cycle of length seven or greater can be bridged by the addition of a chord without creating any triangles. Similarly, in any graph of girth six or greater, any given cycle can be bridged without creating any triangles. There exist graphs of girth four and five, however, that contain cycles of length six that cannot be bridged without creating a triangle. The Petersen graph minus one vertex, as shown in Fig. 1, is such a graph of girth five; replacing one of its vertices with two sharing the same neighbourhood results in a graph of girth four with the same property. To increase the number of cycles in a graph, large chordless cycles can be bridged greedily until the graph is triangle-free but the addition of any edge would create a triangle. This suggests that a cycle-maximal triangle-free graph should contain many cycles of length four or five. Since bipartite graphs are triangle free, complete bipartite graphs and, more specifically, balanced bipartite graphs are natural candidates for maximizing the number of cycles. We verified the following conjecture to be true by exhaustive computer search for $n \leq 13$:

Conjecture 1.1. *The cycle-maximal triangle-free graphs are exactly the bipartite Turán graphs,* $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ *for all n.*

1.1. Overview of results

Our main results, Theorems 4.2 and 5.2, show that Conjecture 1.1 holds for all regular cycle-maximal triangle-free graphs, and all near-regular cycle-maximal triangle-free graphs with greater than 91 vertices. In Section 2 we give some properties of cycle-maximal graphs. In Section 3 we establish bounds on the number of cycles in triangle-free graphs. In Section 4 we prove Theorem 4.2, and in Section 5 we prove Theorem 5.2. Section 6 describes an algorithm for computing the matrix permanent, which is used in our bounds.

1.2. Definitions and notation

Graphs are simple and undirected unless otherwise specified. A *block* in a graph G is a maximal 2-connected subgraph of G. Given a graph G, let $V(G)$, $E(G)$, $\delta(G)$, and $\Delta(G)$ denote, respectively, the vertex set of G, edge set of G, minimum degree