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Note

# A new result on Sylvester's problem ☆

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ABSTRACT

Sylvester's problem, i.e., the problem of the existence of large sets of Kirkman triple systems (LKTSs), is one of the most challenging problems in combinatorial design theory. Most recursive constructions for LKTSs impose resolvable restrictions on the ingredient designs, making the recursions difficult to apply. In this note we provide necessary small designs by using spherical geometries and a direct construction. Finally, we prove the existence of LKTS$(12 \times 7^n - 3)$ for all positive integers $n$. A new result on overlarge sets of Kirkman triple systems is also produced.



## 1. Introduction

In 1850 Kirkman posed the famous 15 schoolgirl problem of arranging fifteen ladies into groups of three over a series of seven days in such a way that no two of them walk twice abreast. In terms of design theory, Kirkman was asking for a Kirkman triple system KTS(15). Soon afterwards, Sylvester raised further questions as for a partition of all the 455 triples from a 15-element set into 13 disjoint KTS(15)s, that is, a large set of KTS(15)s. The search for solutions to the general problem of large sets of Kirkman triple systems (LKTS) then became known as Sylvester's problem. Sylvester's problem aroused great interest among researchers. However, after over 160 years, it is still unsolved; in fact the known results are very limited; see for example a recent survey [2].

Let us begin with some notation and preliminaries. A *t-wise balanced design* (*t*-BD) with parameters $t-(v, K, \lambda)$ is a pair $(X, \mathcal{B})$ where $X$ is a $v$-element set of points and $\mathcal{B}$ is a collection of subsets of $X$ (called *blocks*) with the property that the size of every block is in the set $K$ and every $t$-element subset of $X$ is contained in exactly $\lambda$ blocks. A $t-(v, k, \lambda)$ design is also denoted by $S_\lambda(t, K, v)$. If $\lambda = 1$, the notation $S(t, K, v)$ is usually used and it is called a *Steiner system*. If $K = \{k\}$, we often use the notation $S(t, k, v)$. An $S(2, 3, v)$ is a *Steiner triple system* of order $v$ (or STS$(v)$). An $S(3, 4, v)$ is a *Steiner quadruple system* of order $v$ (or SQS$(v)$).

Let $\mathcal{G}$ be a partition of a set $X$ into subsets called *groups* (or *holes*). A subset $S$ of $X$ is said to be *$\mathcal{G}$-transverse* if $S$ intersects each of the groups in $\mathcal{G}$ in at most one element. A *group divisible t-design* of order $v$ and block sizes from $K$, briefly by GDD$(t, K, v)$, is a triple $(X, \mathcal{G}, \mathcal{B})$ with the following properties:

(1) $X$ is a set of $v$ elements;
(2) $\mathcal{G}$ is a collection of groups which partition $X$;

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(3) $\mathcal{B}$ is a family of $\mathcal{G}$-transverse subsets of $X$ (blocks) each having cardinality in $K$;
(4) every $\mathcal{G}$-transverse $t$-element subset of $X$ is contained in exactly one block.

The *type* of a GDD is defined to be the multiset $T = \{|G| : G \in \mathcal{G}\}$ of group sizes. We also use $a^i b^j c^k \cdots$ to denote the type, which means that in the multiset there are $i$ occurrences of $a$, $j$ occurrences of $b$, etc. A GDD$(t, K, v)$ of type $1^v$ is an S$(t, K, v)$.

Let $(X, \mathcal{G}, \mathcal{B})$ be a GDD. A *parallel class* of $X$ is a collection of blocks in $\mathcal{B}$ that partition $X$. If there is a partition $\Gamma$ of $\mathcal{B}$ into parallel classes, then the GDD is said to be *resolvable* and denoted by RGDD. An RGDD$(t, k, v)$ of type $1^v$ is a resolvable S$(t, k, v)$, denoted by RS$(t, k, v)$. An RS$(2, 3, v)$ is a *Kirkman triple system* of order $v$ (or KTS$(v)$). It is well known that a KTS$(v)$ exists if and only if $v \equiv 3 \pmod 6$ (see [11,13]).

Two STS$(v)$s on the same set $X$ are said to be *disjoint* if they have no triples in common. A partition of all the triples of a $v$-element set into STS$(v)$s, denoted by LSTS$(v)$, is called a *large set* of STS$(v)$s. An LSTS$(v)$ exists if and only if $v \equiv 1, 3 \pmod 6$, $v \neq 7$; see [9,10,15]. A large set of Kirkman triple systems of order $v$, denoted by LKTS$(v)$, is an LSTS$(v)$ in which each STS$(v)$ is a KTS$(v)$.

In this note we study Sylvester's problem and prove the existence of LKTS$(12 \times 7^n - 3)$ for all positive integers $n$. The rest of the paper is organized as follows. In Section 2 we describe a recursive construction for LKTSs. In Section 3 we show the existence of some ingredient designs and prove the new result on Sylvester's problem. Finally in the last section we combine the known constructions to update and expand our results on LKTSs. A new result on overlarge sets of Kirkman triple systems is also produced.

## 2. A recursive construction for LKTSs

In this section we describe a recursive construction for LKTSs. We need the notion of a candelabra system with resolvable derived designs and another structure called an overlarge set of Kirkman frame.

A *candelabra t-system* (as in [12]) of order $v$ with block sizes from $K$, denoted by CS$(t, K, v)$, is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ that satisfies the following properties:

(1) $X$ is a set of $v$ elements (called *points*);
(2) $S$ is a subset (called a *stem*) of $X$ of size $s$;
(3) $\mathcal{G}$ is a collection of subsets (called *groups*) of $X \setminus S$ which partition $X \setminus S$;
(4) $\mathcal{A}$ is a family of subsets (called *blocks*) of $X$, each of cardinality from $K$;
(5) every $t$-element subset $T$ of $X$ with $|T \cap (S \cup G)| < t$ for any $G \in \mathcal{G}$ is contained in exactly one block, and for each $G \in \mathcal{G}$ no $t$-element subsets of $S \cup G$ are contained in any block.

By the *type* of a candelabra system we mean the list $(\{|G| : G \in \mathcal{G}\} : s)$. We also use the exponential notation to denote the type of $\mathcal{G}$ and separate the stem size by a colon. A candelabra system with $t = 3$ and $k = 4$ is a *candelabra quadruple system* and denoted by CQS. Candelabra systems play an important role in the construction of Steiner 3-designs; the reader is referred to [12] for a clear figure and more details.

From an S$(t, k, v)$, by choosing an element $x$, selecting all blocks containing $x$, and deleting $x$ from each, one obtains an S$(t-1, k-1, v-1)$, a *derived design* at the point $x$. An S$(t, k, v)$ *with resolvable derived designs*, abbreviated to RDS$(t, k, v)$, refers to an S$(t, k, v)$ whose derived design at every point is resolvable. An RDSQS$(v)$ denotes an SQS$(v)$ with resolvable derived designs. The collection of all the derived designs of an RDSQS forms a new combinatorial object called the overlarge set (of KTSs). An *overlarge set* of KTS$(v)$s, denoted by OLKTS$(v)$, is a collection $\{(X \setminus \{x\}, \mathcal{B}_x) : x \in X\}$, where $X$ is a $(v+1)$-element set, each $(X \setminus \{x\}, \mathcal{B}_x)$ is a KTS$(v)$ and all the $\mathcal{B}_x$, $x \in X$, form a partition of all triples on $X$. Clearly an RDSQS$(v+1)$ yields an OLKTS$(v)$.

When considering the derived designs of a candelabra system, we need to generalize the concept of $t$-designs to incomplete $t$-designs. An *incomplete t-design* of order $v$ and block size $k$ with a hole of order $h$, denoted by S$(t, k, v; h)$, is a triple $(X, H, \mathcal{A})$, where $X$ is a $v$-element set, $H$ is an $h$-element subset of $X$, and $\mathcal{A}$ is a set of $t$-element subsets of $X$ (blocks), such that every $t$-element subset $T \subseteq X$ with $T \not\subseteq H$ is contained in a unique block and no $t$-element subset of $H$ is contained in any block. If $t = 2$, $k|v$ and $k|h$, we will also use the structure of a resolvable incomplete 2-design RS$(2, k, v; h)$, which is defined to be an S$(2, k, v; h)$ $(X, H, \mathcal{A})$ where the block set $\mathcal{A}$ can be partitioned into $(v-h)/(k-1)$ parallel classes of $X$ and $(h-1)/(k-1)$ holey parallel classes with a hole $H$ (i.e. parallel classes of $X \setminus H$). An incomplete Kirkman triple system KTS$(v, h)$ is an RS$(2, 3, v; h)$.

Suppose that $(X, \emptyset, \mathcal{G}, \mathcal{A})$ is a CS$(3, k+1, gn)$ of type $(g^n : 0)$. (We restrict ourselves to the special case with stem empty; see [2] for the general situation.) For $x \in X$, the derived design $(X \setminus \{x\}, G \setminus \{x\}, \mathcal{A}_x)$ (where $\mathcal{A}_x = \{A \setminus \{x\} : x \in A, A \in \mathcal{A}\}$) clearly forms an incomplete 2-design S$(2, k, gn-1; g-1)$. An RDCS$(3, k+1, gn)$ of type $(g^n : 0)$ denotes a CS$(3, k+1, gn)$ of type $(g^n : 0)$ whose derived design at every point is resolvable. An RDCS$(3, 4, gn)$ of type $(g^n : 0)$ is denoted by RDCQS$(g^n : 0)$.

A *generalized frame* F$(t, k, v\{m\})$ (as in [16]) is a GDD$(t, k, mv)$ of type $m^v$ $(X, \mathcal{G}, \mathcal{B})$ such that the block set $\mathcal{B}$ can be partitioned into subsets $\mathcal{B}_r$, $r \in R$, each $\mathcal{B}_r$ being the block set of a GDD$(t-1, k, m(v-1))$ of type $m^{v-1}$ missing some group $G \in \mathcal{G}$. It is known that an F$(t, k, v\{m\})$ contains $vm/(k-t+1)$ GDD$(t-1, k, m(v-1))$s, i.e., $|R| = vm/(k-t+1)$. Let $R_G = \{r \in R : \mathcal{B}_r \text{ has group set } \mathcal{G} \setminus \{G\}\}$. Then $|R_G| = m/(k-t+1)$.

An F$(2, 3, v\{m\})$ is called a *Kirkman frame*, briefly KF$(m^v)$, and each of its element (i.e. a GDD$(1, 3, m(v-1))$ of type $m^{v-1}$) is a holey parallel class. An F$(3, 3, v\{m\})$ is called an *overlarge set of Kirkman frames* and denoted by OLKF$(m^v)$ if each