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On cyclic regular covers of complete graphs of small order

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ABSTRACT

The paper presents classifications of edge-transitive cyclic regular covers of the complete graphs $\mathbf{K}_5$ and $\mathbf{K}_6$, and arc-transitive cyclic regular covers of the complete graph $\mathbf{K}_7$. Two new infinite families of transitive graphs of valency 4 and 6 are found. As an application, tetravalent edge-transitive graphs of order $5p^2$ with p a prime are classified.

1. Introduction

In this paper, by a graph Γ, we mean a connected, undirected and simple graph with valency at least three.

For a graph Γ, denote its vertex set, edge set, arc set and the full automorphism group by $V\Gamma$, $E\Gamma$, $A\Gamma$ and $\mathrm{Aut}\,\Gamma$, respectively. For a vertex v, let $\Gamma(v)$ denote the vertices which are adjacent to v, and let X be an automorphism group of Γ, that is, $X \le \mathrm{Aut}\,\Gamma$. If X is transitive on $V\Gamma$, $E\Gamma$ or $A\Gamma$, then Γ is called *X-vertex-transitive*, *X-edge-transitive* or *X-arc-transitive*, respectively. A 2-arc of Γ is three distinct vertices (u, v, w) with u, w both adjacent to v. Then Γ is called *$(X, 2)$-arc-transitive* if X is transitive on the set of all 2-arcs of Γ. If X acts transitively on $V\Gamma$ and $E\Gamma$ but not on $A\Gamma$, then Γ is called *X-half-transitive*.

A transitive permutation group is called *quasiprimitive* if each of its nontrivial normal subgroups is transitive, while it is called *bi-quasiprimitive* if each of its minimal normal subgroups has at most two orbits and there exists one with exactly two orbits. It is well known that each edge-transitive graph is a multi-cover of a 'basic graph': vertex quasiprimitive or vertex bi-quasiprimitive edge-transitive graph. In particular, a remarkable theorem of Praeger [23] shows that each 2-arc-transitive graph is a regular cover of a basic 2-arc-transitive graph (this result has been slightly generalized to the locally-primitive graph case in [14]). Upon these reasons, characterizing regular covers of transitive graphs has received much attention in the literature. For example, [3,15,18,17] established some basic theory of cover theory, which has been successfully applied to classify cyclic or elementary abelian regular covers of a number of symmetric graphs of small valency, including the Peterson graph [19], the Heawood graph [17], the Möbius–Kantor graph [16], the complete bipartite graph $\mathbf{K}_{3,3}$ [6], the Pappus graph [21], the octahedron graph [13] and the 3-dimensional cube graph [8]. Moreover, 2-arc-transitive cyclic, $\mathbb{Z}_p^2$ and $\mathbb{Z}_p^3$ regular covers with p a prime of complete graphs are determined in [5,4]; arc-transitive cyclic and elementary abelian covers of the complete graph $\mathbf{K}_4$ are presented in [6]; and arc-transitive elementary abelian covers of the complete graph $\mathbf{K}_5$ are obtained in [12]. In the present paper, we classify edge-transitive cyclic regular covers of the complete graphs $\mathbf{K}_5$ and $\mathbf{K}_6$, and arc-transitive cyclic regular covers of the complete graph $\mathbf{K}_7$.

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Throughout the paper, for a positive integer n, we denote by $\mathbb{Z}_n$ the cyclic group of order n with additive notation for its operation. For an element a of a group, denote by $o(a)$ the order of a. For two groups N and H, denote by $N \cdot H$ an extension of N by H, and if such an extension is split, then we write $N : H$ instead of $N \cdot H$.

The following theorem classifies edge-transitive cyclic regular covers of $\mathbf{K}_5$ and $\mathbf{K}_6$. For convenience, see definitions of regular cover, multi-cover and fibre-preserving group in Section 2.

Theorem 1.1. *Let Γ be a $\mathbb{Z}_n$-regular cover of the complete graph $\Sigma := \mathbf{K}_5$ or $\mathbf{K}_6$. Suppose that the fibre-preserving group X acts edge-transitively on Γ. Then either*

(1) *$\Sigma = \mathbf{K}_5$, and one of the following holds:*
  (i) *$\Gamma = \mathrm{CC}(n, 5; k, l)$, as in Example 2.4, is X-half-transitive, where $\langle k, l\rangle = \mathbb{Z}_n$, and $l \neq ks$ with $s^2 = -1$;*
  (ii) *$\Gamma = \mathrm{CC}(n, 5; 1, s)$ is X-arc-transitive but not $(X, 2)$-arc-transitive, where $s^2 = -1$ and $n \neq 2$;*
  (iii) *$\Gamma = \mathbf{K}_{5,5} - 5\mathbf{K}_2$ is 2-arc-transitive; or*

(2) *$\Sigma = \mathbf{K}_6$, and $\Gamma = \mathbf{K}_{6,6} - 6\mathbf{K}_2$ is 2-arc-transitive.*

A graph Γ is called a *Cayley graph* of a group G if there is a subset $S \subseteq G \setminus \{1\}$, with $S = S^{-1} := \{g^{-1} \mid g \in S\}$, such that $V\Gamma = G$, and two vertices g and h are adjacent if and only if $hg^{-1} \in S$. We denote this Cayley graph by $\mathrm{Cay}(G, S)$. It is well known that a graph Γ is isomorphic to a Cayley graph of a group G if and only if $\mathrm{Aut}\,\Gamma$ contains a subgroup which is isomorphic to G and acts regularly on $V\Gamma$, see [1, Proposition 16.3]. For convenience, we often write this regular group as G. If G is normal in X with $X \leq \mathrm{Aut}\,\Gamma$, then Γ is called an *X-normal Cayley graph*.

As an application of Theorem 1.1, the next theorem classifies tetravalent edge-transitive graphs of order $5p^2$ with p a prime. Since such graphs for the case $p \leq 5$ are given in [26], we here only list such graphs with $p \geq 7$. We notice that cubic arc-transitive graphs of order $4p$, $6p$, $4p^2$ or $6p^2$ are classified in [6]; cubic arc-transitive graphs of order $8p$ or $8p^2$ are determined in [7]; tetravalent half-arc-transitive graphs of order $4p$ and $2p^2$ are characterized in [9,25], respectively.

Theorem 1.2. *Let Γ be a tetravalent X-edge-transitive graph of order $5p^2$, where $X \leq \mathrm{Aut}\,\Gamma$ and $p \geq 7$ is a prime. Then Γ is an X-normal Cayley graph, and one of the following is true.*

(1) *$\Gamma = \mathrm{CC}(p^2, 5; k, l)$ with $\langle k, l\rangle \cong \mathbb{Z}_{p^2}$, as in Example 2.4, is a $\mathbb{Z}_{p^2}$-regular cover of $\mathbf{K}_5$;*

(2) *Γ is a $\mathbb{Z}_p^2$-regular cover of $\mathbf{K}_5$, listed in row 2 of Table 1 in [12, Theorem 2.1];*

(3) *Γ is a multi-cover of the cycle $\mathbf{C}_5$ of length 5, $X_v \leq \mathbb{Z}_2^2$ with $v \in V\Gamma$, and one of the following holds.*
  (i) *$\Gamma = \mathrm{Cay}(\langle a\rangle, \{a, a^{-1}, a^{i+1}, a^{-i-1}\})$ with $o(a) = 5p^2$ and $5 \mid i$,*
  $\Gamma = \mathrm{Cay}(\langle a\rangle, \{a^p, a^{-p}, a^{5j+p}, a^{-5j-p}\})$ with $o(a) = 5p^2$ and $p \nmid j$, or
  $\Gamma = \mathrm{Cay}(\langle a\rangle, \{a^{p^2}, a^{-p^2}, a^{5k+p^2}, a^{-5k-p^2}\})$ with $o(a) = 5p^2$ and $p \nmid k$;
  (ii) *$\Gamma = \mathrm{Cay}(G, \{a, a^{-1}, ab, a^{-1}b^{-1}\})$, where $G \cong \mathbb{Z}_p^2 \times \mathbb{Z}_5$, $a, b \in G$ such that $o(a) = 5p$, $o(b) = p$ and $b \notin \langle a\rangle$;*
  (iii) *$\Gamma = \mathrm{Cay}(G, \{a, a^{-1}, ab, (ab)^{-1}\})$, where G is nonabelian, a is not a p-element, b is a p-element such that $\langle a, b\rangle = G$, and there is an involution $\sigma \in \mathrm{Aut}\,(G)$ such that $a^\sigma = ab$.*

Graphs in Theorem 1.2 are explicitly determined with the only exception of part (3)(iii). We remark that, by analysing each of the non-isomorphic nonabelian groups of order $5p^2$ (there are a few such non-isomorphic groups), graphs in part (3)(iii) may be more specifically characterized by a very long list similar to part (3)(i). For convenience, we omit this complicated and direct analysis.

The arc-transitive cyclic regular covers of $\mathbf{K}_7$ are classified in the following theorem.

Theorem 1.3. *Let Γ be a $\mathbb{Z}_n$-regular cover of the complete graph $\mathbf{K}_7$. Suppose that the fibre-preserving group X acts arc-transitively on Γ. Then one of the following is true.*

(1) *$\Gamma = \mathrm{CC}(n, 7; k, l, s)$ with $n \geq 3$, as in Example 2.5, is X-arc-transitive;*

(2) *$\Gamma = \mathbf{K}_{7,7} - 7\mathbf{K}_2$ is 2-arc-transitive.*

This paper is organized as follows. After this introduction, we give some preliminary results and new examples in Section 2. Then, Theorems 1.1–1.3 are proved in Section 3.

2. Preliminaries and examples

In this section, we present certain preliminary results and construct two infinite families of examples appearing in Theorems 1.1–1.3.

For two graphs Γ and Σ, Γ is called a *cover* (or *covering*) of Σ with a projection ρ if ρ is a surjection from $V\Gamma$ to $V\Sigma$ such that the restriction $\rho|_{\Gamma(\tilde{v})} : \Gamma(\tilde{v}) \to \Gamma(v)$ is a bijection for each $v \in V\Sigma$ and each preimage $\tilde{v}$ of v under ρ. Further, Γ is called a *regular cover* (or *K-regular cover*) if there is a semiregular subgroup $K \leq \mathrm{Aut}\,\Gamma$ such that Σ is isomorphic to the quotient graph Γ_K, say by ϕ, and the quotient map $\Gamma \to \Gamma_K$ is the composition $\rho\phi$. If K is cyclic, then Γ is called a *cyclic regular cover* of Σ. We call that Γ is a *multi-cover* of a quotient graph Γ_N with $N \leq \mathrm{Aut}\,\Gamma$ if it has the property that u^N and $v^N \in V\Gamma_N$ are adjacent in Γ_N if and only if the induced subgraph $[u^N, v^N]$ of Γ is isomorphic to $k\mathbf{K}_2$, where k is independent to the choices of u, v, refer to [20, p. 169]. For each vertex $v \in V\Sigma$, the set of preimages of v under ρ, denoted by $\rho^{-1}(v)$, is called a *fibre*. An automorphism of Γ is called *fibre-preserving* if it maps each fibre to a fibre. The group, consisting of all