Discrete Mathematics 331 (2014) 43–52

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The partition dimension of strong product graphs and Cartesian product graphs

Ismael González Yero [a], Marko Jakovac [b,*], Dorota Kuziak [c], Andrej Taranenko [b]

[a] Departamento de Matemáticas, Escuela Politécnica Superior de Algeciras, Universidad de Cádiz, Av. Ramón Puyol s/n, 11202 Algeciras, Spain

[b] Faculty of Natural Sciences and Mathematics, University of Maribor, Koroška 160, 2000 Maribor, Slovenia

[c] Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Av. Països Catalans 26, 43007 Tarragona, Spain

ARTICLE INFO

Article history:
Received 3 August 2013
Received in revised form 11 April 2014
Accepted 23 April 2014
Available online 20 May 2014

Keywords:
Resolving partition
Partition dimension
Strong product graphs
Cartesian product graphs
Graphs partitioning

ABSTRACT

Let $G = (V, E)$ be a connected graph. The distance between two vertices $u, v \in V$, denoted by $d(u, v)$, is the length of a shortest u, v-path in G. The distance between a vertex $v \in V$ and a subset $P \subset V$ is defined as $\min\{d(v, x) : x \in P\}$, and it is denoted by $d(v, P)$. An ordered partition $\{P_1, P_2, \ldots, P_t\}$ of vertices of a graph G, is a resolving partition of G, if all the distance vectors $(d(v, P_1), d(v, P_2), \ldots, d(v, P_t))$ are different. The partition dimension of G is the minimum number of sets in any resolving partition of G. In this article we study the partition dimension of strong product graphs and Cartesian product graphs. Specifically, we prove that the partition dimension of the strong product of graphs is bounded below by four and above by the product of the partition dimensions of the factor graphs. Also, we give the exact value of the partition dimension of strong product graphs when one factor is a complete graph and the other one is a path or a cycle. For the case of Cartesian product graphs, we show that its partition dimension is less than or equal to the sum of the partition dimensions of the factor graphs minus one. Moreover, we obtain an upper bound on the partition dimension of Cartesian product graphs, when one factor is a complete graph.

1. Introduction

The idea of a partition dimension was introduced by Chartrand et al. in [6] to gain more insight about another closely related graph parameter called the metric dimension of a graph. The partition dimension of graphs is also studied in [3,7,16,17]. Given a connected graph $G = (V, E)$ and an ordered partition $\Pi = \{P_1, P_2, \ldots, P_t\}$ of the vertices of G, the *partition representation* of a vertex $v \in V$ with respect to the partition Π is the vector $r(v|\Pi) = (d(v, P_1), d(v, P_2), \ldots, d(v, P_t))$, where $d(v, P_i)$, with $1 \le i \le t$, represents the distance between the vertex v and the set P_i, that is $d(v, P_i) = \min_{u \in P_i}\{d(v, u)\}$ ($d(v, u)$ denotes the distance between the vertices v and u). We say that Π is a *resolving partition* of G if for every pair of distinct vertices $u, v \in V$, $r(u|\Pi) \ne r(v|\Pi)$. The *partition dimension* of G is the minimum number of sets in any resolving partition of G and is denoted by $pd(G)$.

The concepts of resolvability and location in graphs were described independently by Harary and Melter [8], and Slater [15], to define the same structure in a graph. After these papers were published several authors developed diverse

* Corresponding author.
E-mail addresses: ismael.gonzalez@uca.es (I. González Yero), marko.jakovac@uni-mb.si (M. Jakovac), dorota.kuziak@urv.cat (D. Kuziak), andrej.taranenko@uni-mb.si (A. Taranenko).

http://dx.doi.org/10.1016/j.disc.2014.04.026

theoretical works about this topic [2,1,3–7,12]. Slater described the usefulness of these ideas into long range aids to navigation [15]. Also, these concepts have some applications in chemistry for representing chemical compounds [10,11] or in problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [13]. Other applications of this concept to navigation of robots in networks and other areas appear in [4,9,12].

Given a connected graph $G = (V, E)$ and an ordered set of vertices $S = \{v_1, v_2, \ldots, v_k\}$ of G, the *metric representation* of a vertex $v \in V$ with respect to S is the vector $r(v|S) = (d(v, v_1), \ldots, d(v, v_k))$. We say that S is a *resolving set* of G if for every pair of distinct vertices $u, v \in V$, $r(u|S) \neq r(v|S)$. The *metric dimension* of G is the minimum cardinality of any resolving set of G, and it is denoted by $\dim(G)$. The metric dimension of graphs is studied in [2,1,3–5,16].

It is natural to think that the partition dimension and metric dimension are related; in [6] it was shown that for any nontrivial connected graph G we have

$$pd(G) \leq \dim(G) + 1. \tag{1}$$

We recall that the strong product of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph $G \boxtimes H = (V, E)$, such that $V = \{(a, b) : a \in V_1,\ b \in V_2\}$ and two vertices $(a, b) \in V$ and $(c, d) \in V$ are adjacent in $G \boxtimes H$ if and only if, either

- $a = c$ and $bd \in E_2$, or
- $b = d$ and $ac \in E_1$, or
- $ac \in E_1$ and $bd \in E_2$.

Also, the Cartesian product of G and H is the graph $G\Box H = (V, E)$, such that $V = \{(a, b) : a \in V_1,\ b \in V_2\}$ and two vertices $(a, b) \in V$ and $(c, d) \in V$ are adjacent in $G \boxtimes H$ if and only if, either

- $a = c$ and $bd \in E_2$, or
- $b = d$ and $ac \in E_1$.

Let $v \in V_2$. We refer to the set $V_1 \times \{v\}$ as a G-layer. Similarly $\{u\} \times V_2$, $u \in V_1$ is an H-layer. When referring to a specific G or H layer, we denote them by G^v or ${}^uH$, respectively. Layers can also be regarded as the graphs induced on these sets. Obviously, a G-layer or H-layer is isomorphic to G or H, respectively.

Studies about partition dimension in product graphs have been presented in [17], where the authors obtained some results about the partition dimension of the Cartesian product graphs. Also in [14] several results about the partition dimension of corona product graphs were presented. In this article we begin with the study of the partition dimension of strong product graphs and we also continue with the study of the partition dimension of Cartesian product graphs.

2. Strong product graphs

We begin with the following useful lemmas.

Lemma 1. *Let G and H be two connected graphs and let A and B be two proper subsets of vertices of G and H, respectively. If $a \in A$ and $b \notin B$, then*

$$d_{G\boxtimes H}((a, b), A \times B) = \min_{v\in B}\{d_H(b, v)\}.$$

Equivalently, if $a \notin A$ and $b \in B$, then

$$d_{G\boxtimes H}((a, b), A \times B) = \min_{u\in A}\{d_G(a, u)\}.$$

Proof. Suppose $a \in A$ and $b \notin B$. We first prove that for every $v \in B$, $d_{G\boxtimes H}((a, b), A \times \{v\}) = d_H(b, v)$.

$$\begin{aligned} d_{G\boxtimes H}((a, b), A \times \{v\}) &= \min_{(u,v)\in A\times\{v\}}\{d_{G\boxtimes H}((a, b), (u, v))\} \\ &= \min_{u\in A}\{d_{G\boxtimes H}((a, b), (u, v))\} \\ &= \min_{u\in A}\{\max\{d_G(a, u), d_H(b, v)\}\} \\ &= \min_{u\in A}\{\max_{a=u}\{d_G(a, u), d_H(b, v)\}, \max_{a\neq u}\{d_G(a, u), d_H(b, v)\}\} \\ &= \min_{u\in A}\{d_H(b, v), \max_{a\neq u}\{d_G(a, u), d_H(b, v)\}\} \\ &= d_H(b, v). \end{aligned}$$

Thus we obtain that $d_{G\boxtimes H}((a, b), A \times B) = \min_{v\in B}\{d_{G\boxtimes H}((a, b), A \times \{v\})\} = \min_{v\in B}\{d_H(b, v)\}$. Analogously we prove that if $a \notin A$ and $b \in B$, then $d_{G\boxtimes H}((a, b), A \times B) = \min_{u\in A}\{d_G(a, u)\}$ and the proof is complete. $\Box$

Lemma 2 ([6]). *Let G be a connected graph of order $n \geq 2$. Then $pd(G) = 2$ if and only if G is a path graph.*

The following straightforward claim is useful to prove our next results.

Claim 3. *Let G and H be two connected non-trivial graphs. If there exists a resolving partition of $G \boxtimes H$ with exactly three sets, say $\Pi = \{A, B, C\}$, then there exists no subgraph of $G \boxtimes H$ isomorphic to K_4, such that it contains at least one vertex from each of the sets A, Band C.*