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Note

Cutwidth of triangular grids☆

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ABSTRACT

When the vertices of an n-vertex graph G are numbered by the integers 1 through n, the *length* of an edge is the difference between the numbers on its endpoints. Two edges *overlap* if the larger of their lower numbers is less than the smaller of their upper numbers. The *bandwidth* of G is the minimum, over all numberings, of the maximum length of an edge. The *cutwidth* of G is the minimum, over all numberings, of the maximum number of pairwise overlapping edges. The bandwidth of triangular grids was determined by Hochberg, McDiarmid, and Saks in 1995. We show that the cutwidth of the triangular grid with side-length l is $2l$.

1. Introduction

The bandwidth and cutwidth problems for graphs are optimization problems applied in VLSI design, network communications, and other areas involving the graph layout (see [3,4,6,14]). We compute the cutwidth for certain special graphs; first we formulate the problem.

Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G, respectively. A *numbering* (or *labeling*) of an n-vertex graph G is a bijection $f\colon V(G) \to \{1, \ldots, n\}$. Given a numbering f of G, let

$$c(G, f) = \max_{1\le i<n} \left|\{uv \in E(G)\colon f(u) \le i < f(v)\}\right|.$$

When f is viewed as embedding G in a path, $c(G, f)$ is the maximum number of pairwise overlapping edges, measuring congestion. The *cutwidth* of G, denoted by $c(G)$, is $\min\{c(G, f)\colon f \text{ is a numbering of } G\}$. A numbering f that minimizes $c(G, f)$ is *optimal*.

Similarly, for a given numbering f, let

$$B(G, f) = \max\{|f(u) - f(v)|\colon uv \in E(G)\}.$$

When f is viewed as embedding G in a path, $B(G, f)$ is the maximum length (dilation) of an edge. The *bandwidth* of G, denoted by $B(G)$, is $\min\{B(G, f)\colon f \text{ is a numbering of } G\}$.

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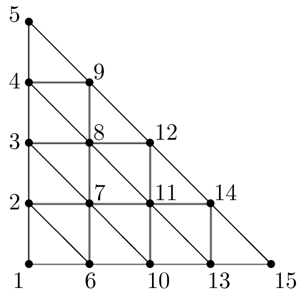

Fig. 1. T_4, with an optimal numbering.

Much work has been done on computing bandwidth and cutwidth of special graphs, especially graphs relevant in the areas of application. Let P_n and C_n denote the path and cycle with n vertices, respectively. The *Cartesian product* of graphs G and H, denoted by $G\square H$, is the graph with vertex set $V(G) \times V(H)$ in which two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. The early results include

- $B(P_m\square P_n) = \min\{m, n\}$ for $m, n \geq 2$.
- $B(P_m\square C_n) = \min\{2m, n\}$ for $m \geq 2$, $n \geq 3$.
- $B(C_m\square C_n) = 2\min\{m, n\} - \delta_{m,n}$ for $m, n \geq 3$,

where $\delta_{m,n} = 1$ if $m = n$ and $\delta_{m,n} = 0$ otherwise. The first two of these results were obtained by Chvátalová [5]; the third was obtained by Li, Tao, and Shen [10].

The cutwidth of these graphs has also been computed [13,16]:

- $c(P_m\square P_n) = \min\{m, n\} + 1$ for $m \geq 2$, $n \geq 3$.
- $c(P_m\square C_n) = \min\{2m, n + 1\} + 1$ for $m, n \geq 3$.
- $c(C_m\square C_n) = 2\min\{m, n\} + 2$ for $m, n \geq 3$.

In these examples, always $B(G) < c(G)$. Equality holds when $G = C_n$. Only some sparse graphs (such as some trees [12]) are known to satisfy $c(G) \leq B(G)$; it would be interesting to determine which graphs satisfy this inequality.

The cutwidth was also computed for "meshes" [17]. Polynomial-time algorithms are known for computing cutwidth on trees [18] and for recognizing when $c(G) \leq k$ [7,15]. An exact formula for cutwidth on trees called "iterated caterpillars" appears in [12]. The cutwidth of n-dimensional hypercubes was studied in [1,11].

The *triangular grid* T_l is the graph whose vertices are the nonnegative integer triples with sum l such that vertices (x, y, z) and (x', y', z') are adjacent if and only if $|x - x'| + |y - y'| + |z - z'| = 2$. That is, two vertices are adjacent when they agree in one coordinate and differ by 1 in the other two coordinates. M.L. Weaver and the third author conjectured $B(T_l) = l + 1$. Using topological methods (Sperner's Lemma), Hochberg, McDiarmid, and Saks [9] proved this as a special case of a more general result computing the bandwidth of a family of triangulations of planar discs.

Our main result is as follows.

Theorem 1.1. $c(T_l) = 2l$.

Similar arguments yield the cutwidths of rectangular grids with added diagonal edges.

2. Preliminaries

Our graphs have no loops or multi-edges. For a set S of vertices in a graph G, let $\overline{S} = V(G) - S$. The *neighborhood* of S, denoted by $N(S)$, is $\{v \in \overline{S}\colon uv \in E(G)$ for some $u \in S\}$; note that $N(S) \subseteq \overline{S}$. The *boundary* of S is $N(\overline{S})$ (some authors refer to $N(S)$ as the boundary of S). The *coboundary* of S, denoted by $\partial(S)$ following the notation of [2], is the set of edges in G that have endpoints in both S and $\overline{S}$.

The definitions yield an immediate rephrasing of $c(G, f)$.

Observation 2.1. *If f is a numbering of a graph G, and $S_i = \{v \in V(G)\colon f(v) \leq i\}$, then $c(G, f) = \max_{1\leq i<n} |\partial(S_i)|$.*

We may draw T_l in the plane by putting vertex (x, y, z) at point (x, y) and using (x, y) as the name of the vertex. Now the vertex set is the set of nonnegative integer pairs with sum at most l, and two vertices (x, y) and (x', y') are adjacent if and only if (a) $|x - x'| + |y - y'| = 1$ or (b) $|x - x'| + |y - y'| = 2$ and $x + y = x' + y'$. Fig. 1 shows T_4 drawn in this way.

With this embedding, the edges of T_l lie in three sets of parallel lines, yielding three partitions of $V(T_l)$. The *horizontal sets* or *rows* have second coordinate fixed; let $P_j = \{(x, j) \in V(T_l)\colon 0 \leq x \leq l - j\}$. The *vertical sets* or *columns* have first coordinate fixed; let $Q_i = \{(i, y) \in V(T_l)\colon 0 \leq y \leq l - i\}$. The *slanted sets* or *diagonals* have the sum of the two coordinates fixed; let $R_k = \{(x, y) \in V(T_l)\colon x + y = l - k\}$. Each edge of T_l joins two vertices in one of these sets; the edges accordingly are *horizontal*, *vertical*, or *slanted* edges, respectively.

Lemma 2.2. $c(T_l) \leq 2l$.