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Decomposition of complete bipartite graphs into paths and cycles

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ABSTRACT

We give conditions for decomposability of the complete bipartite graph $K_{m,n}$ into paths and cycles having k edges. In particular, we find necessary and sufficient conditions for such decomposition in $K_{m,n}$, when $m \geq \frac{k}{2}$, $n \geq \lceil \frac{k+1}{2} \rceil$ for $k \equiv 0 \pmod 4$ and when $m, n \geq 2k$ for $k \equiv 2 \pmod 4$. As a consequence, we show that for nonnegative integers p and q, an even integer k, and odd n with $n > 4k$, there exists a decomposition of the complete graph K_n into p paths and q cycles both having k edges if and only if $k(p+q) = \binom{n}{2}$ and $p \neq 1$.

1. Introduction

Unless stated otherwise all graphs considered here are finite, simple, and undirected. For the standard graph-theoretic terminology the readers are referred to [6]. Let P_k, C_k, K_k respectively denote a path, cycle, and complete graph on k vertices, and let $K_{m,n}$ denote the complete bipartite graph with m and n vertices in the parts. A graph whose vertex set is partitioned into sets $V_1, \ldots, V_m$ such that the edge set is $\bigcup_{i \neq j \in [m]} V_i \times V_j$ is a *complete m-partite graph*, denoted by $K_{n_1,\ldots,n_m}$ when $|V_i| = n_i$ for all i. For any integer $\lambda > 0$, λG denotes the graph consisting of λ edge-disjoint copies of G. The *λ-multiplication* of G, denoted $G(\lambda)$, is the multigraph obtained from a graph G by replacing each edge with λ edges. For two graphs G and H, their *lexicographic product* or *wreath product* $G \otimes H$ has vertex set $V(G) \times V(H)$ with two vertices (g_1, h_1) and (g_2, h_2) adjacent whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$. The complement of the graph G is denoted by $\overline{G}$.

By a *decomposition* of a graph G, we mean a list of edge-disjoint subgraphs of G whose union is G (ignoring isolated vertices). When there is such a decomposition using copies of (not necessarily distinct) graphs $H_1, \ldots, H_k$, we say that $H_1, \ldots, H_k$ *decompose* G. When each subgraph in a decomposition is isomorphic to H, we say that G has an *H-decomposition*. If G has a decomposition into p copies of H_1 and q copies of H_2, then we say that G has a *$\{pH_1, qH_2\}$-decomposition*. If such a decomposition exists for all values of p and q satisfying trivial necessary conditions, then we say that G has a *$\{H_1, H_2\}_{\{p,q\}}$-decomposition* or G is *fully $\{H_1, H_2\}$-decomposable*.

Study of $\{H_1, H_2\}_{\{p,q\}}$-decomposition for K_n and $K_{m,n}$ is not new. Abueida, Daven, and Roblee [1,3] completely determined the values of n for which $K_n(\lambda)$ admits a $\{pH_1, qH_2\}$-decomposition such that $H_1 \cup H_2 \cong K_t$, when $\lambda \geq 1$ and $|V(H_1)| = |V(H_2)| = t$, where $t \in \{4, 5\}$. Let S_k denotes a star on k vertices, i.e. $S_k = K_{1,k-1}$. Abueida and Daven [2] proved that there exists a $\{pK_k, qS_{k+1}\}$-decomposition of K_n for $k \geq 3$ and $n \equiv 0, 1 \pmod k$. Abueida and O'Neil [4] proved that for $k \in \{3, 4, 5\}$, there exists a $\{pC_k, qS_k\}$-decomposition of λK_n whenever $n \geq k + 1$ except for the ordered triples $(k, n, \lambda) \in \{(3, 4, 1), (4, 5, 1), (5, 6, 1), (5, 6, 2), (5, 6, 4), (5, 7, 1), (5, 8, 1)\}$. Shyu [13] obtained a necessary and sufficient condition

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on (p, q) for the existence of $\{P_5, C_4\}_{\{p,q\}}$-decomposition of K_n. Shyu [14] proved that K_n has a $\{P_4, S_4\}_{\{p,q\}}$-decomposition if and only if $n \geq 6$ and $3(p+q) = \binom{n}{2}$. He also proved that K_n has a $\{P_k, S_k\}_{\{p,q\}}$-decomposition with a restriction $p \geq k/2$ when k even (respectively, $p \geq k$ when k odd). Shyu [15] proved that K_n has a $\{C_4, S_5\}_{\{p,q\}}$-decomposition if and only if $4(p+q) = \binom{n}{2}$, $q \neq 1$ if n is odd and $q \geq \max\{3, \lceil \frac{n}{4} \rceil\}$ if n is even.

Chou et al. [8] proved that for a given triple (p, q, r) of nonnegative integers, G decomposes into p copies of C_4, q copies of C_6, and r copies of C_8 such that $4p+6q+8r = |E(G)|$ in the following two cases: (a) $G = K_{m,n}$ with m and n both even at least 4, except $K_{4,4}$, (b) G is obtained from $K_{n,n}$ with n odd by deleting a perfect matching. Chou and Fu [7] proved that the existence of $\{C_4, C_{2t}\}_{\{p,q\}}$-decomposition of $K_{2u,2v}$, where $t/2 \leq u, v < t$ when t even (respectively, $(t+1)/2 \leq u, v \leq (3t-1)/2$, when t odd) implies such decomposition in $K_{2m,2n}$, where $m, n \geq t$ (respectively, $m, n \geq (3t+1)/2$). Jeevadoss and Muthusamy [9] reduced the bounds in the sufficient conditions obtained by Chou and Fu [7] for the existence of $\{C_4, C_{2t}\}_{\{p,q\}}$-decomposition of $K_{2m,2n}$, when $t > 2$. Lee et al. [11,10] obtained a necessary and sufficient condition on (p, q) for the existence of $\{P_k, S_k\}_{\{p,q\}}$-decomposition of $K_{n,n}$ and $K_{m,n}$. Shyu [16] proved that $K_{m,n}$ has a $\{P_k, S_k\}_{\{p,q\}}$-decomposition for some m and n. He also obtained a necessary and sufficient condition on (p, q) for the existence of $\{P_4, S_4\}_{\{p,q\}}$-decomposition of $K_{m,n}$.

In this paper, we study only the existence of $\{P_{k+1}, C_k\}_{\{p,q\}}$-decomposition of $K_{m,n}$ and K_n, so we abbreviate the notation as $(k; p, q)$-*decomposition*. The obvious necessary condition for such existence is $k(p+q) = |E(G)|$. We only consider cases where all vertices are of even degree, in which the case $p \neq 1$ is also obviously necessary, since the endpoints of a single path in the decomposition would have to have odd degree. Call the situation with $k(p+q) = |E(G)|$, all vertex degrees are even, and $p \neq 1$ the *good case*. We prove that in the good case $K_{m,n}$ has a $(k; p, q)$-decomposition in the following two situations: (i) $m \geq k/2$ and $n \geq \lceil \frac{k+1}{2} \rceil$ when $k \equiv 0 \pmod 4$ and (ii) $m, n \geq 2k$ when $k \equiv 2 \pmod 4$. Also, when $k \equiv 2 \pmod 4$, we show that if $K_{m,n}$ has a $(k; p, q)$-decomposition in the good case with $k/2 \leq m, n < k$, then such decompositions also exist in the good case when $k/2 \leq m < 2k$ and $n \geq k$. Finally, we apply these results to show that in the good case K_n has a $(k; p, q)$-decomposition if k is even and $n > 4k$.

Let $K_{n,n}$ be the complete bipartite graph with bipartition (X, Y), where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. For $0 \leq i \leq n-1$, let $F_i(X, Y)$ denote the set $\{x_j y_{j+i} : j \in [n]\}$, where subscripts are treated modulo n. Clearly $F_i(X, Y)$ is a 1-factor of $K_{n,n}$, called the 1-factor of *displacement* i. Also, $\bigcup_{i=0}^{n-1} F_i(X, Y) = K_{n,n}$.

Remark. (i) For $n \in \mathbb{N}$, let $K_{2n,2n}$ have partite sets $X_1 \cup X_3$ and $X_2 \cup X_4$, where $X_r = \{x_1^r, \ldots, x_n^r\}$. For $0 \leq i \leq n-1$, let $F_i(X_r, X_s) = \{x_j^r x_{j+i}^s : j \in [n]\}$, where arithmetic in subscripts is taken modulo n. Note that the union of the sets $F_i(X_r, X_s)$ over all i and $(r, s) \in \{(1, 2), (2, 3), (3, 4), (4, 1)\}$ decomposes $K_{2n,2n}$.
(ii) For $k \in \{0, \ldots, n-1\}$ and $i \in \{1, 2, 3, 4\}$, the set $F_k(X_i, X_{i+1}) \cup F_{k+1}(X_i, X_{i+1})$ forms a 2-regular subgraph of $K_{2n,2n}$ consisting a cycle of length $2n$.
(iii) For any positive integer n, the set $F_0(X_1, X_2) \cup F_0(X_2, X_3) \cup F_0(X_3, X_4) \cup F_1(X_4, X_1)$ forms a Hamilton cycle of $K_{2n,2n}$.
(iv) $F_k(X_i, X_j) = F_{n-k}(X_j, X_i)$, where $0 \leq k \leq n-1$.

To prove our results, we state the following:

Theorem 1.1 (*Truszczynski [19]*). *If $k, m, n \in \mathbb{N}$ with m, n even and $m \geq n$, then $K_{m,n}$ has a P_{k+1}-decomposition if and only if $m \geq \lceil \frac{k+1}{2} \rceil$; $n \geq \lceil \frac{k}{2} \rceil$ and $mn \equiv 0 \pmod k$.*

Theorem 1.2 (*Sotteau [17]*). *If $k, m, n \in \mathbb{N}$ with $k \geq 2$, then $K_{m,n}$ has a C_{2k}-decomposition if and only if the obvious necessary conditions hold (all degrees even and $2k \mid mn$).*

Theorem 1.3 (*Shyu [13]*). *Let k, t, p be positive even integers such that $k \geq 4$ and $t > p$. If $k \leq 2(t-p)$ and $k \leq 2p$, then there exists a $\{pP_{k+1}, (t-p)C_k\}$-decomposition of $K_{k,t}$.*

2. $(k; p, q)$-decomposition of $K_{m,n}$ when $k \equiv 0 \pmod 4$

In this section we investigate the existence of $(k; p, q)$-decomposition of the complete bipartite graph $K_{m,n}$ when $k \equiv 0 \pmod 4$. Let $x_0 x_1 \ldots x_{k-2} x_{k-1}$ and $(x_0 x_1 \ldots x_{k-1} x_0)$ respectively denote the path P_k and the cycle C_k with vertices $x_0, x_1, \ldots, x_{k-1}$ and edges $x_0x_1, x_1x_2, \ldots, x_{k-2}x_{k-1}, x_{k-1}x_0$.

Construction. *Let $\mathbb{C}_\lambda$ and $\mathbb{C}_\mu$ be two cycles of length k, where $\mathbb{C}_\lambda = (x_1, x_2, \ldots, x_k, x_1)$ and $\mathbb{C}_\mu = (y_1, y_2, \ldots, y_k, y_1)$. If v is a common vertex of $\mathbb{C}_\lambda$ and $\mathbb{C}_\mu$ such that at least one neighbour of v from each cycle (say, x_i and y_j) does not belong to the other cycle then we have two edge-disjoint paths of length k, say $\mathbb{P}_\lambda$ and $\mathbb{P}_\mu$ from $\mathbb{C}_\lambda$ and $\mathbb{C}_\mu$ as follows (see Fig. 1):*

$$\mathbb{P}_\lambda = (\mathbb{C}_\lambda - vx_i) \cup vy_j, \qquad \mathbb{P}_\mu = \left(\mathbb{C}_\mu - vy_j\right) \cup vx_i.$$

Remark. Let k, m_1, m_2, n_1, n_2 be positive even integers such that $m_1 \leq n_1$ and $m_2 \leq n_2$. If K_{m_1,n_1} and K_{m_2,n_2} have a $(k; p, q)$-decomposition then $K_{m_1,n_1} \cup K_{m_2,n_2}$ has a such decomposition.

Lemma 2.1. *If k, m, n be positive even integers such that $k \equiv 0 \pmod 4$, $k/2 \leq m \leq n < k$ and $n \neq k/2$, then the graph $K_{m,n}$ has a $(k; p, q)$-decomposition.*