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Hamilton decompositions of 6-regular Cayley graphs on even Abelian groups with involution-free connections sets

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ABSTRACT

Alspach conjectured that every connected Cayley graph on a finite Abelian group A is Hamilton-decomposable. Liu has shown that for $|A|$ even, if $S = \{s_1, \ldots, s_k\} \subset A$ is an inverse-free strongly minimal generating set of A, then the Cayley graph Cay$(A; S^\star)$, is decomposable into k Hamilton cycles, where $S^\star$ denotes the inverse-closure of S. Extending these techniques and restricting to the 6-regular case, this article relaxes the constraint of strong minimality on S to require only that S be strongly a-minimal, for some $a \in S$ and the index of $\langle a \rangle$ be at least four. Strong a-minimality means that $2s \notin \langle a \rangle$ for all $s \in S \setminus \{a, -a\}$. Some infinite families of open cases for the 6-regular Cayley graphs on even order Abelian groups are resolved. In particular, if $|s_1| \geq |s_2| > 2|s_3|$, then Cay$(A; \{s_1, s_2, s_3\}^\star)$ is Hamilton-decomposable.

1. Introduction

In 1969, Lovász [1] posed the following problem: "Let us construct a finite, connected, undirected graph which is [vertex-transitive] and has no simple path containing all vertices". From this problem, the following conjecture was posed: every finite, connected vertex-transitive graph has a Hamilton path. The conjecture has generated a vast and rich body of literature, yet there exist no known examples of connected vertex-transitive graphs which do not possess a Hamilton path. Furthermore, only four non-trivial connected vertex-transitive graphs are known which do not possess a Hamilton *cycle* (Bondy [6]). Even more interesting is the fact that none of these four are Cayley graphs. Ergo, it has been widely conjectured that connected Cayley graphs on three or more vertices have Hamilton cycles. This conjecture too is unresolved, though it has been verified for finite Abelian groups (this is believed to be folklore, one proof is due to Marušič [17]). If A is a finite Abelian group, and S is a subset of $A - \{0\}$, that is inverse-closed, i.e., $s \in S \Leftrightarrow -s \in S$, then the *Cayley graph* of A with *connection set* S, is the graph X, denoted $X = \text{Cay}(A; S)$, with $V(X) = A$ and $E(X) = \{\{x, y\}: y - x \in S\}$. In fact, connected Cayley graphs of Abelian groups are rich with Hamilton cycles (see Chen–Quimpo [8]) and Alspach [2] conjectured in 1984 that this class of graphs is Hamilton-decomposable. Alspach's conjecture remains open in general (see the survey by Curran–Gallian [9]) despite being investigated from many vantage points: the valency, the group order, the group type, restrictions on the connection set, etc. The following theorem is a summary of complete results when only the valency is considered:

Theorem 1.1 (*Alspach et al. [4], Bermond et al. [5], Dean [11], Fan et al. [12], Liu [14,15], Westlund et al. [19]*). *Every connected k-regular Cayley graph on a finite Abelian group A is Hamilton-decomposable if* $k \leq 5$ *or* $k = 6$ *and* $|A|$ *is odd.*

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2. Preliminaries

A path (resp. cycle) that spans the vertices of a graph is called a *Hamilton path* (resp. *Hamilton cycle* or *HC*). A *Hamilton decomposition* of a graph is a partition of its edge set into Hamilton cycles if it has even valency, or Hamilton cycles and a single perfect matching if it has odd valency. A graph is *Hamilton-decomposable* (or *HD*) if it admits a Hamilton decomposition.

The Cayley graph $X = \mathrm{Cay}(A; S)$ is connected if and only if S is a generating set for A. If $y - x = s \in S$, then $\{x, y\}$ is *generated by s*, or $\{x, y\}$ is an *s-edge*. The subgraph Y of X is *generated by s* if $E(Y)$ consists of all s-edges of $E(X)$. If s is an involution of A, i.e., $s \neq 0$ and $s = -s$, then the subgraph generated by s is a 1-factor of X. If s is a non-involution, then the subgraph generated by s is a 2-factor of X.

Let $S \subset A$. S is *inverse-free* if $s \in S$ implies either $s = -s$ or $-s \notin S$. S is *involution-free* if no element of S is an involution. The *inverse-closure* of S, denoted $S^\star$, is the smallest superset of S that is inverse-closed. S is *minimal* if for all $s \in S$, the subgroup generated by the elements of $S \setminus \{s, -s\}$ does not contain s; and *strongly minimal* if for all $s \in S$, the subgroup generated by $S \setminus \{s, -s\}$ does not contain $2s$. Clearly, any strongly-minimal subset of a group and any minimal subset of an odd order group is involution-free.

Liu has proved some general results on Alspach's conjecture when restrictions on the connection set are considered.

Theorem 2.1 (*Liu [15,16]*). *Let A be an Abelian group and S an inverse-free subset of $A - \{0\}$ that generates A. The Cayley graph $Cay(A; S^\star)$, is HD if either one of the following holds:*

1. *The order of A is odd, and S is minimal.*
2. *The order of A is even, at least four, and S is strongly minimal.*

Theorem 2.2 (*[16]*). *Let A be an Abelian group of even order at least four, and $S \subset A$ be an inverse-free minimal generating set. If $\langle s\rangle$ has odd index for all $s \in S$, then $Cay(A; S^\star)$ is HD.*

Alspach, Caliskan, and Kreher [3] have recently made significant advances in outlining new approaches to obtaining Hamilton decompositions via liftings, with a focus on Cayley graphs of odd order. This manuscript only examines the specific case of 6-regular even order Cayley graphs on finite Abelian groups. The following results apply to 6-regular Cayley graphs.

Theorem 2.3 (*[12]*). *If A is an Abelian group of odd order generated by $S = \{s_1, s_2, s_3\} \subset A - \{0\}$, which is inverse-free and involution-free, such that $|s_1| \geq |s_2| > |s_3|$, then $Cay(A; S^\star)$ is HD.*

Theorem 2.4 (*Dean [10,11]*). *Let A be a cyclic group generated by $S = \{s_1, s_2, s_3\} \subset A - \{0\}$, where S is inverse-free and involution-free. $Cay(A; \{s_1, s_2, s_3\}^\star)$ is HD if either one of the following holds:*

1. *A has odd order.*
2. *A has even order, and $A = \langle s_3\rangle$.*

Theorem 2.5 (*Westlund [18]*). *Let A be an Abelian group of even order generated by $S = \{s_1, s_2\} \subset A - \{0\}$, where S is involution-free and inverse-free. If there exists an $s_3 \in A$, such that $\langle s_3\rangle$ has index two in A, then $Cay(A; (S \cup \{s_3\})^\star)$ is HD.*

3. New results

This article relaxes the strong minimality requirement of Theorem 2.1(2) in the 6-regular case. The following definition will be used throughout:

Definition 3.1. A subset $S \subseteq A$ is *strongly a-minimal* for some fixed $a \in S$, if $2s \notin \langle a\rangle$ for all $s \in S \setminus \{a, -a\}$.

Clearly, if S is strongly minimal, then S is strongly a-minimal for all $a \in S$. The converse is not true. For example, $S = \{(1, 0), (0, 1), (3, 1)\} \subset \mathbb{Z}_4 \oplus \mathbb{Z}_4$ is strongly a-minimal for all $a \in S$, yet S is not even minimal.

Given a subgroup $B \leq A$, and $a \in A$, use $\overline{a}$ to denote the coset $a + B$ in A, and $|A : B|$ the index of B in A. Cayley graphs of quotient groups have been used to find Hamilton cycles. If $X = \mathrm{Cay}(A; S^\star)$, and $B \leq A$, then $\overline{X} = \mathrm{Cay}(A/B; \overline{S}^\star)$, where $\overline{S}^\star = \{\overline{s}: s \in S^\star\}$, is called the *quotient* (multi)graph of X by B (see Fig. 1). The next proposition follows immediately from the definitions so far.

Proposition 3.2. *If A is an Abelian group, and $S = \{s_1, \ldots, s_k\} \subset A - \{0\}$ is a generating set for A that is both inverse-free and involution-free, then $Cay(A; S^\star)$ is $2k$-regular and connected. Furthermore, if S is strongly a-minimal, for some $a \in S$, then $Cay(A/\langle a\rangle; \overline{S}^\star)$ is $2(k-1)$-regular.*

The focus of this manuscript is on Cayley graphs $\mathrm{Cay}(A; S^\star)$, where $S = \{s_1, s_2, s_3\} \subset A - \{0\}$ and S is both involution-free and inverse-free. By Proposition 3.2 $\mathrm{Cay}(A; S^\star)$ is 6-regular, and each element of S generates a 2-factor. Note, that if S is not involution-free, then $|S^\star| \leq 5$, and so $X = \mathrm{Cay}(A; S^\star)$ is k-regular for some $k \in \{1, 2, 3, 4, 5\}$. By Theorem 1.1 X is HD. The following theorem, proved in [12], forms the basis for a portion of the proof of the Main Result.

Theorem 3.3 (*[12]*). *Let A be an Abelian group and $S = \{s_1, s_2, s_3\} \subset A - \{0\}$ be an inverse-free and involution-free generating set for A. If S is strongly s_3-minimal, where $|s_3|$ is odd, and $|A : \langle s_3\rangle| \geq 9$, then $Cay(A; S^\star)$ is HD.*