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A fourth extremal even unimodular lattice of dimension 48

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ABSTRACT

We show that there is a unique extremal even unimodular lattice of dimension 48 which has an automorphism of order 5 of type 5-(8, 16)-8. Since the three known extremal lattices do not admit such an automorphism, this provides a new example of an extremal even unimodular lattice in dimension 48.

1. Introduction

A lattice L in Euclidean space $(\mathbb{R}^n, (\,,\,))$ is a free $\mathbb{Z}$-module of rank n containing a basis of $\mathbb{R}^n$. The lattice L is called *even*, if the associated quadratic form $Q : \mathbb{R}^n \to \mathbb{R}_{\geq 0}, x \mapsto Q(x) := \frac{1}{2}(x,x)$ is integral on L, so $Q(L) \subseteq \mathbb{Z}$. Then L is contained in its dual lattice

$$L^{\#} := \{y \in \mathbb{R}^n \mid (y, \ell) \in \mathbb{Z} \text{ for all } \ell \in L\}.$$

L is *unimodular*, if $L = L^{\#}$. Any lattice L defines a sphere packing, whose density measures its error correcting properties. One of the main goals in lattice theory is to find dense lattices. This is a very difficult problem, the densest lattices are known only in dimension $n \leq 8$ and in dimension 24 [3], for $n = 8$ and $n = 24$ the densest lattices are even unimodular lattices. The density of a unimodular lattice is measured by its *minimum*,

$$\min(L) := \min\{2Q(\ell) \mid 0 \neq \ell \in L\}.$$

For even unimodular lattices the theory of modular forms allows one to bound this minimum $\min(L) \leq 2 + 2\lfloor \frac{n}{24} \rfloor$ and *extremal lattices* are those even unimodular lattices L that achieve equality. Of particular interest are extremal even unimodular lattices L in the *jump dimensions* $24m$. For $m = 1$ there is a unique extremal even unimodular lattice, the *Leech lattice*, which is the densest 24-dimensional lattice [3]. By [4], its automorphism group is a covering group of the sporadic simple Conway group Co_1. The 196 560 minimal vectors of the Leech lattice form the unique tight spherical 11-design and realize the maximal kissing number in dimension 24. In dimension 72 one knows one extremal unimodular lattice [9]. The existence of such a lattice was a longstanding open problem. In dimension 48 there are at least four extremal even unimodular lattices. They are the densest known lattices in their dimension and realize the maximal known kissing number 52,416,000. It is a very interesting problem to classify all 48-dimensional extremal even unimodular lattices. To get an idea of how many such lattices might exist, I started a program to find all extremal lattices L whose automorphism group

$$\mathrm{Aut}(L) = \{g \in \mathrm{GL}(L) \mid Q(xg) = Q(x) \text{ for all } x\}$$

is not too small. In [10] I classified all 48-dimensional extremal lattices that have an automorphism of order a whose Euler phi value is $\varphi(a) > 24$. All these lattices are isometric to one of the lattices P_{48p}, P_{48q}, or P_{48n}, which were known before. The present paper classifies all extremal lattices invariant under a certain automorphism of order 5. It turns out that there is a unique such lattice, P_{48m}, and this lattice is not isometric to one of the lattices above (see Table 1).

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Table 1
The known extremal even unimodular lattices in the jump dimensions.

Name	Autom. group	Order	Ref.
Λ_{24}	$2.Co_1$	$8315553613086720000 = 2^{22}3^9 5^4 7^2 11\ 13\ 23$	[7,4]
P_{48p}	$(\mathrm{SL}_2(23) \times S_3) : 2$	$72864 = 2^5 3^2 11\ 23$	[5,10]
P_{48q}	$\mathrm{SL}_2(47)$	$103776 = 2^5 3\ 23\ 47$	[5,10]
P_{48n}	$(\mathrm{SL}_2(13) \mathrm{Y} \mathrm{SL}_2(5)).2^2$	$524160 = 2^7 3^2 5\ 7\ 13$	[8,10]
P_{48m}	$(C_5 \times C_5 \times C_3) : (D_8 \mathrm{Y} C_4)$	$1200 = 2^4 3\ 5^2$	This paper
Γ_{72}	$(\mathrm{SL}_2(25) \times \mathrm{PSL}_2(7)) : 2$	$5241600 = 2^8 3^2 5^2 7\ 13$	[9,10]

Table 2
The possible types of automorphisms $\sigma \neq -1$ of prime order.

Type	$L_K(\sigma)$	$L_I(\sigma)$	Example	Complete
47-(1, 2)-1	Unique	Unique	P_{48q}	[10, Theorem 5.6]
23-(2, 4)-2	Unique	At least 2	P_{48q}, P_{48p}	
13-(4, 0)-0	{0}	At least 1	P_{48n}	
11-(4, 8)-4	Unique	At least 1	P_{48p}	
7-(8, 0)-0	{0}	At least 1	P_{48n}	
7-(7, 6)-5	$\sqrt{7}A_6^{\#}$	Not known	Not known	
5-(12, 0)-0	{0}	At least 2	P_{48n}, P_{48m}	
5-(10, 8)-8	$\sqrt{5}E_8$	At least 1	P_{48m}	
5-(8, 16)-8	$[2.\mathrm{Alt}_{10}]_{16}$	Λ_{32}	P_{48m}	Theorem 3.2
3-(24, 0)-0	{0}	At least 3	$P_{48p}, P_{48n}, P_{48m}$	
3-(20, 8)-8	$\sqrt{3}E_8$	Not known	Not known	
3-(16, 16)-16	$\sqrt{3}(E_8 \perp E_8)$	At least 4	$P_{48p}, P_{48q}, P_{48n}$	
3-(16, 16)-16	$\sqrt{3}D_{16}^{+}$	At least 4	Not known	
3-(15, 18)-15	Unique	Two	Not known	
3-(14, 20)-14	?	Unique	Not known	
3-(13, 22)-13	?	Unique	Not known	
2-(24, 24)-24	$\sqrt{2}\Lambda_{24}$	$\sqrt{2}\Lambda_{24}$	P_{48n}	
2-(24, 24)-24	$\sqrt{2}O_{24}$	$\sqrt{2}O_{24}$	$P_{48n}, P_{48p}, P_{48m}$	

2. The type of an automorphism

The notion of the type of an automorphism of a lattice L was introduced in [10]. It was motivated by the analogous notion of a type of an automorphism of a code.

Let $\sigma \in \mathrm{GL}_n(\mathbb{Q})$ be an element of prime order p. Let $K := \ker(\sigma - 1)$ and $I := \mathrm{im}(\sigma - 1)$. Then K is the fixed space of σ and the action of σ on I gives rise to a vector space structure on I over the p-th cyclotomic number field $\mathbb{Q}[\zeta_p]$. In particular $n = d + z(p-1)$, where $d := \dim_{\mathbb{Q}}(K)$ and $z = \dim_{\mathbb{Q}[\zeta_p]}(I)$.

If L is a σ-invariant $\mathbb{Z}$-lattice, then L contains a sublattice M with

$$L \geq M = (L \cap K) \oplus (L \cap I) =: L_K(\sigma) \oplus L_I(\sigma) \geq pL$$

of finite index $[L : M] = p^s$ where $s \leq \min(d, z)$.

Definition 2.1. The triple $p - (z, d) - s$ is called the *type* of the element $\sigma \in \mathrm{GL}(L)$.

Remark 2.2. Let (L, Q) be an even unimodular lattice and $\sigma \in \mathrm{Aut}(L)$ be of type $p - (z, d) - s$. Then $L_I^{\#}(\sigma)/L_I(\sigma) \cong (\mathbb{Z}[\zeta_p]/(1-\zeta_p))^s$ and $L_K^{\#}(\sigma)/L_K(\sigma) \cong (\mathbb{Z}/p\mathbb{Z})^s$ as $\mathbb{Z}[\sigma]$-modules. In particular $0 \leq s \leq \min(z, d)$. If $z = s$ then $(1-\sigma)L_I^{\#}(\sigma) = L_I(\sigma)$ and hence $(L_I^{\#}(\sigma), pQ) = (X, \mathrm{trace}_{\mathbb{Q}[\zeta_p]/\mathbb{Q}}(h))$ is the trace lattice of a Hermitian unimodular $\mathbb{Z}[\zeta_p]$ lattice (X, h) of rank $\dim_{\mathbb{Z}[\zeta_p]}(X) = z$.

Remark 2.3. If L is an even lattice and p is odd, then $L_K(\sigma)$ and $L_I(\sigma)$ are also even lattices, because $L_K(\sigma) \oplus L_I(\sigma)$ is a sublattice of odd index in L.

In [10] we narrowed down the possible types of prime order automorphisms of an extremal even unimodular lattice in dimension 48. By Remark 2.3 the fixed lattice of an element of order 3 cannot be $\sqrt{3}D_{12}^{+}$, as this is an odd lattice. So Type 3-(18, 12)-12 is not possible and the possible types are among the ones in Table 2.

Remark 2.4. Table 2 lists the possible types of prime order automorphisms $\sigma \neq -1$. The type usually determines the genus of the $\mathbb{Z}$-lattice $L_K(\sigma)$ and the $\mathbb{Z}[\zeta_p]$-lattice $L_I(\sigma)$. If these genera are either classified in the literature (in particular the unimodular genera) or easily computed in Magma, we give the names or the number of lattices of minimum ≥ 6 in these genera. Column "example" lists the known examples and the last column gives the two instances where the classification of the lattices is known to be complete.