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Chromatic numbers of copoint graphs of convex geometries

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ABSTRACT

We study the copoint graph of a convex geometry. We give a family of copoint graphs for which the ratio of the chromatic number to the clique number can be arbitrarily large. For any natural numbers $1 < d < k$, we study the existence of a number $K_d(k)$ so that the chromatic number of the copoint graph of a convex geometry on a set of at least $K_d(k)$ elements, with every d-element subset closed, has chromatic number at least k. Our results are analogues of results of Erdős and Szekeres for convex geometries realizable by point sets in R^d, where cliques in the copoint graph correspond to subsets of points in convex position.

1. Introduction

Let X be a finite set. An *alignment* $\mathscr{L}$ is a collection of subsets of X such that $\emptyset \in \mathscr{L}, X \in \mathscr{L}$, and if $A, B \in \mathscr{L}$ then $A \cap B \in \mathscr{L}$. A set $C \subseteq X$ is *closed* or *convex* if $C \in \mathscr{L}$. Following Edelman and Jamison [5], we also view $\mathscr{L}$ as a closure operator on the subsets of X, where $\mathscr{L}(A) = \bigcap\{C : C \text{ is closed and } A \subseteq C\}$. The closure operator $\mathscr{L}$ is *anti-exchange* if for any $x, y \notin \mathscr{L}(C)$, $x \in \mathscr{L}(C \cup y)$, then $y \notin \mathscr{L}(C \cup x)$. Equivalently, for any closed set C, with $C \neq X$, there is at least one closed set of the form $C \cup p$ for $p \notin C$. A pair $(X, \mathscr{L})$ where $\mathscr{L}$ is an anti-exchange closure operator is called a *convex geometry*. The closed sets of a convex geometry $(X, \mathscr{L})$ can be partially ordered by inclusion to form a lattice, $L_{\mathscr{L}}$. A subset $A \subseteq X$ is *convexly independent* or *independent* if for all $p \in A$, $p \notin \mathscr{L}(A - p)$.

A set $C \in \mathscr{L}$ is a *copoint* if it is maximal in $X - p$ for some $p \in X$. If C is a copoint, there is exactly one set in $\mathscr{L}$ of the form $C \cup p$ for $p \notin C$. The unique p is denoted $\alpha(C)$, and we say that the copoint C is *attached* to $\alpha(C)$. We will sometimes refer to a copoint C by the pair $(\alpha(C), C)$. The *copoint graph* of $(X, \mathscr{L})$, $\mathcal{G}(X, \mathscr{L})$, has as its vertex set the set of copoints of $(X, \mathscr{L})$, with copoints C and D adjacent if and only if $\alpha(C) \in D$ and $\alpha(D) \in C$. The definition of independent sets shows that a set $A \subseteq X$ is independent in $(X, \mathscr{L})$, if and only if there is a clique in $\mathcal{G}(X, \mathscr{L})$ of copoints attached to the elements of A. Thus the clique number of $\mathcal{G}(X, \mathscr{L})$ equals the size of the largest independent set of $(X, \mathscr{L})$.

If X is a set of points in $\mathbb{R}^d$, and $\mathscr{L} = \{C \subseteq X : X \cap \text{conv}(C) = C\}$, then $(X, \mathscr{L})$ is a convex geometry, called the convex geometry *realized* by X. One can show that if the points of a set X are in $\mathbb{R}^d$, then a set $A \subseteq X$ is the vertex set of a convex polytope if and only if A is independent in $(X, \mathscr{L})$. For point sets $X \subseteq \mathbb{R}^2$ in *general position*, that is no three points are on the same line, there is a famous conjecture of Erdős and Szekeres [8] that X contains the vertex set of a convex n-gon whenever $|X| > 2^{n-2}$. Morris [16] proved that for a point set X in general position in $\mathbb{R}^2$, the *chromatic* number of $\mathcal{G}(X, \mathscr{L})$ is at least n whenever $|X| > 2^{n-2}$. This result highlights the need to understand the relationship between the chromatic number and clique number of copoint graphs. We will present several results involving the clique and chromatic number of copoint

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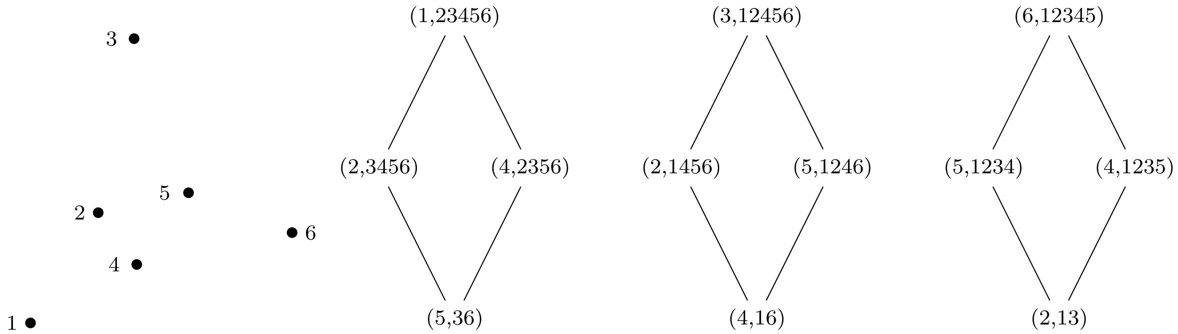

Fig. 1. A six point set and its poset of copoints.

graphs for general convex geometries. One should keep in mind, however, that convex geometries realized by point sets in $\mathbb{R}^d$ form a small subset of the set of all convex geometries.

In Section 2 we answer a question posed by Beagley [2], giving a family of convex geometries $([n], \mathscr{L}_{d,n})$, for positive integers $d < n$, for which $\omega(\mathcal{G}(X, \mathscr{L})) = d + 1$ and $\chi(\mathcal{G}(X, \mathscr{L})) \geq \lceil \log_2(n+1) \rceil$. This shows that the chromatic number of $\mathcal{G}(X, \mathscr{L})$ cannot be bounded by a function of the clique number of $\mathcal{G}(X, \mathscr{L})$. The convex geometry $([n], \mathscr{L}_{d,n})$ will have the property that it is *d-free*, i.e. $\mathscr{L}$ will contain every d-element subset of $[n]$. The fact that every d-element subset is closed, together with the alignment property implies that every k-element subset is closed for every integer k with $0 \leq k \leq d$. This is a property that is satisfied by convex geometries realized by point sets in general position in $\mathbb{R}^d$.

In Section 3 we investigate the effect that the d-free property alone will have on the chromatic number of $\mathcal{G}(X, \mathscr{L})$. The first main result we prove is that if $1 < d < k$ there exists a number $K_d(k)$ so that any d-free convex geometry on a set of size at least $K_d(k)$ will have the chromatic number of its copoint graph at least k. We also show that $K_d(d+2) = d + 3$ for all $d > 1$, analogous to a result of Esther Klein that every set of 5 planar points in general position contains the vertex set of a convex 4-gon.

To close this introductory section, we give the smallest set of points in the plane in general position for which $\omega(\mathcal{G}(X, \mathscr{L}))$ and $\chi(\mathcal{G}(X, \mathscr{L}))$ differ.

Of the 16 order types of 6 planar points in general position [1], there is only one with this property. It is given in Fig. 1. The copoints are shown to the right of the point set, in the form $(\alpha(C), C)$ where $\alpha(C)$ is the point to which copoint C is attached. The copoints are partially ordered by set containment. The subgraph of $\mathcal{G}(X, \mathscr{L})$ induced by the copoints of $(X, \mathscr{L})$ of size bigger than 3 forms the complement of a 9-cycle. This graph has chromatic number 5 and clique number 4. From the figure one can see that the point set does not contain the vertex set of a convex 5-gon, so the clique number of the whole graph is 4.

2. Construction of a convex geometry

Beagley [2] asked the following question: Is $\chi(\mathcal{G}(X, \mathscr{L}))/\omega(\mathcal{G}(X, \mathscr{L})) \leq c$ for some constant c? We construct a family of convex geometries indexed by integers d, n with clique number of $d + 1$ and chromatic number at least $\lceil \log_2(n+1) \rceil$.

Let n be a positive integer and $\{1, 2, \ldots, n\} = [n]$. When $i = 0$, then $[i] = \emptyset$. Let d be a positive integer, $d < n$, and define $\mathscr{L}_{d,n} = \{([i] \cup J) | 0 \leq i \leq n, J \subseteq \{i+2, \ldots, n\}, |J| \leq d\}$.

Proposition 2.1. *For n, d positive integers with $d < n$, the pair $([n], \mathscr{L}_{d,n})$ is a d-free convex geometry.*

Proof. It is easy to see that $\mathscr{L}_{d,n}$ is closed under intersection and $\emptyset, [n] \in \mathscr{L}_{d,n}$. Let C be in $\mathscr{L}_{d,n}$, $C \neq [n]$. If $C = [i] \cup J$ with $0 \leq i \leq n, J \subseteq \{i+2, \ldots, n\}, |J| \leq d$, then $C \cup \{i+1\} \in \mathscr{L}_{d,n}$, so $([n], \mathscr{L}_{d,n})$ is a convex geometry. To see that $([n], \mathscr{L}_{d,n})$ is d-free, note that if $|J| \leq d$ and i is the smallest element of $[n] \setminus J$, then $J = [i-1] \cup J'$ where $|J'| \leq d$. □

For each $i \in \{1, 2, \ldots, n-d\}$, define $A_i = \{[i-1] \cup J | J \subseteq \{i+1, i+2, \ldots, n\}, |J| = d\}$ and for each $i \in \{n-d+1, n-d+2, \ldots, n\}$ let $A_i = \{[i-1] \cup \{i+1, i+2, \ldots, n\}\}$.

Proposition 2.2. *For $i = 1, 2, \ldots, n$, A_i is the set of copoints of $([n], \mathscr{L}_{d,n})$ attached to i.*

Proof. Suppose $C \in A_i$. Then $C \in \mathscr{L}_{d,n}$ and $i \notin C$. If $j \notin C \cup \{i\}$ then it must be true that $i \in \{1, 2, \ldots, n-d\}$. In that case $|(C \cup \{j\}) \cap \{i+1, i+2, \ldots, n\}| > d$, so $C \cup \{j\} \notin \mathscr{L}_{d,n}$. It follows that C is a copoint attached to i. Suppose $C \in \mathscr{L}_{d,n}$ with $i \notin C$. If $[i-1] \not\subseteq C$ then $C \cup \{j\} \in \mathscr{L}_{d,n}$ for j the smallest element of $[i-1] \setminus C$. Thus C is not a copoint attached to i. If $[i-1] \subseteq C$ and $C \cap \{i+1, i+2, \ldots, n\}$ has fewer than d elements and is not $\{i+1, i+2, \ldots, n\}$, then there exists $j > i$ so that $C \cup \{j\} \in \mathscr{L}_{d,n}$. Again C is not a copoint attached to i. Thus every copoint of $\mathscr{L}_{d,n}$ attached to i is in A_i. □

We define the graph $G_{d,n}$ to be the copoint graph $\mathcal{G}([n], \mathscr{L}_{d,n})$. The size of the maximum clique in $G_{d,n}$ can be found using the size of the largest independent set.