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Rainbow numbers for matchings in plane triangulations

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ABSTRACT

Given two graphs G and H, let $f(G, H)$ denote the maximum number c for which there is a way to color the edges of G with c colors such that every subgraph H of G has at least two edges of the same color. Equivalently, any edge-coloring of G with at least $rb(G, H) = f(G, H) + 1$ colors contains a rainbow copy of H, where a rainbow subgraph of an edge-colored graph is such that no two edges of it have the same color. The number $rb(G, H)$ is called the *rainbow number* of H *with respect* to G. If G is a complete graph K_n, the numbers $f(K_n, H)$ and $rb(K_n, H)$ are called *anti-Ramsey numbers* and *rainbow numbers*, respectively.

In this paper we will study the existence of rainbow matchings in plane triangulations. Denote by kK_2 a matching of size k and $\mathcal{T}_n$ the class of all plane triangulations of order n. The *rainbow number* $rb(\mathcal{T}_n, kK_2)$ is the minimum number of colors c such that, if $kK_2 \subseteq T_n \in \mathcal{T}_n$, then any edge-coloring of T_n with at least c colors contains a rainbow copy of kK_2. We give lower and upper bounds on $rb(\mathcal{T}_n, kK_2)$ for all $k \geq 3$ and $n \geq 2k$. Furthermore, we obtain the exact values of $rb(\mathcal{T}_n, kK_2)$ for $2 \leq k \leq 4$ and $n \geq 2k$.

1. Introduction

We use [1] for terminology and notation not defined here and consider finite and simple graphs only. For a vertex $v \in V(G)$, let $N_G(v)$ denote the set of vertices adjacent to the vertex v in G. If G is edge colored in a given way and a graph $H \subseteq G$ contains no two edges of the same color, H is called a *rainbow subgraph* of G or, in other words, a *rainbow* (copy of) H. On the other hand, if all edges of H are colored with the same color, H is called *monochromatic*. Let $f(G, H)$ denote the maximum number of colors in an edge-coloring of G with no rainbow copy of H. The number $f(K_n, H)$ is called *anti-Ramsey number* and has been introduced by Erdős, Simonovits and Sós in [4] (and denoted there by $f(n, H)$). It is closely related to the *rainbow number* $rb(G, H)$ representing the minimum number c of colors such that any edge-coloring of G with at least c colors contains a rainbow subgraph isomorphic to H. Evidently, $rb(G, H) = f(G, H) + 1$.

A recent survey concerning rainbow numbers is given in [6].

In 2004, Schiermeyer [10] determined the rainbow numbers $rb(K_n, kK_2)$ for all $k \geq 2$ and $n \geq 3k + 3$, where kK_2 is a matching M of size k. And the rainbow numbers $rb(K_n, kK_2)$ have been computed step by step in [2,5,10].

Theorem 1.1.

$$rb(K_n, kK_2) = \begin{cases} 4, & n = 4 \text{ and } k = 2; \\ ext(n, (k-1))K_2 + 3, & n = 2k \text{ and } k \geq 7; \\ ext(n, (k-1))K_2 + 2, & \text{otherwise} \end{cases}$$

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where $ext(n, kK_2)$ *is the maximum number of edges that a graph G of order n can have with no subgraph isomorphic to* kK_2 *and* $ext(n, kK_2) = \max\{\binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1)\}$ *(determined by Erdős and Gallai [3]).*

The main focus of this paper is to consider the analogous problem for matchings when the host graph G is a plane triangulation. A plane triangulation is a connected planar graph which can be drawn in the plane so that every face is a triangle. Thus, if T_n is a plane triangulation of order $n \geq 4$, then $|E(T_n)| = 3n - 6$ and $\delta(T_n) \geq 3$.

Let $\mathcal{T}_n$ denote the class of all plane triangulations of order n. We denote by $rb(\mathcal{T}_n, H)$ the minimum number of colors c such that, if $H \subseteq T_n \in \mathcal{T}_n$, then any edge-coloring of T_n with at least c colors contains a rainbow copy of H. Recently, rainbow numbers for cycles in plane triangulations have been determined in [7].

In this paper, we give lower and upper bounds on $rb(\mathcal{T}_n, kK_2)$ for all $k \geq 3$ and $n \geq 2k$. Furthermore, we obtain the exact values of $rb(\mathcal{T}_n, kK_2)$ for $2 \leq k \leq 4$ and $n \geq 2k$.

2. The lower bounds

In this section, we give lower bounds on $rb(\mathcal{T}_n, kK_2)$ for all $k \geq 3$ and $n \geq 2k$.

Lemma 2.1.

$$rb(\mathcal{T}_n, 3K_2) \geq n + 1 \quad \textit{for all } n \geq 6.$$

Proof. Let T_n be a plane triangulation with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = \{v_1v_2\} \cup \{v_1v_i, v_2v_i | 3 \leq i \leq n\} \cup \{v_iv_{i+1} | 3 \leq i \leq n-1\}$. Now color all edges v_1v_i for $2 \leq i \leq n$ with $n-1$ distinct colors and all remaining edges with one extra color. Now observe that T_n does not contain a rainbow $3K_2$. ■

We will show that the lower bound can be achieved for all $n \geq 7$ and $k = 3$, and thus obtain the exact value of $rb(\mathcal{T}_n, 3K_2)$ for all $n \geq 7$ in the next section. Furthermore, we will also show that $rb(\mathcal{T}_6, 3K_2) = 8$.

Lemma 2.2.

$$rb(\mathcal{T}_n, kK_2) \geq 2n + 2k - 9 \quad \textit{for all } k \geq 4 \textit{ and } n \geq 2k.$$

Proof. Let T_n be a plane triangulation of order n with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = \{v_1v_2\} \cup \{v_1v_i, v_2v_i | 3 \leq i \leq n\} \cup \{v_iv_{i+1} | 3 \leq i \leq n-1\}$. Now color all edges v_iv_{i+1} for $2k - 5 \leq i \leq n - 1$ with color 1 and all remaining $(3n - 6) - (n - (2k - 5)) = 2n + 2k - 11$ edges with $2n + 2k - 11$ distinct colors $2, 3, \dots, 2n + 2k - 10$. Now observe that T_n does not contain a rainbow kK_2. ■

We will show that the lower bound can be achieved for all $n \geq 8$ and $k = 4$, and thus obtain the exact value of $rb(\mathcal{T}_n, 4K_2)$ for all $n \geq 8$ in the next section.

3. $rb(\mathcal{T}_n, 2K_2)$, $rb(\mathcal{T}_n, 3K_2)$ and $rb(\mathcal{T}_n, 4K_2)$

In this section, we will give the exact values of $rb(\mathcal{T}_n, kK_2)$ for $2 \leq k \leq 4$ and $n \geq 2k$. First, we provide some additional notations. Given a graph G and $X, Y \subseteq V(G)$, we denote by $E_G(X, Y)$ the set of edges which have exactly one endvertex in X and one in Y. We also denote $E_G(X, X)$ by $E_G(X)$.

Theorem 3.1.

$$rb(\mathcal{T}_n, 2K_2) = \begin{cases} 4, & n = 4; \\ 2, & n \geq 5. \end{cases}$$

Proof. For $n = 4$ the edges of K_4 can be partitioned into three $2K_2$. Coloring the edges of each $2K_2$ with a distinct color shows that $rb(\mathcal{T}_n, 2K_2) \geq 4$. Using four colors, by the pigeonhole principle, there is a $2K_2$ such that its edges are colored distinct. This proves that $rb(\mathcal{T}_4, 2K_2) = 4$.

For $n \geq 5$ let G be a plane triangulation whose edges are colored with two colors 1 and 2. Since $n \geq 5$, then $\delta(G) \geq 3$. Let $w \in V(G)$ be a vertex whose incident edges contain both colors. Let $N_G(w) = \{v_1, v_2, \dots, v_d\}$ for $d \geq 3$. Since G is a plane triangulation, then $G[N_G[w]]$ contains a wheel (a cycle with an additional vertex being adjacent to each vertex of the cycle). If $d = 3$, then $G[N_G[w]] \cong K_4$. Since $n \geq 5$, there is an edge outside $G[N_G[w]]$ incident with a vertex from $N_G[w]$. We assume that $uv_3 \in E(G)$ for a vertex $u \notin N_G[w]$. Suppose G contains no rainbow $2K_2$. Thus, $c(uv_3) = c(wv_1) = c(wv_2) = c(v_1v_2)$ and $c(wv_3) = c(v_1v_2)$. This contradicts to the fact that the incident edges of w contain both colors. Hence, we may assume $d \geq 4$.

Let $v_1, v_2, \dots, v_d$ be the rim vertices and assume that $c(wv_1) = 1$, $c(wv_2) = 2$. Then $wv_1, v_{d-1}v_d$ or $wv_2, v_{d-1}v_d$ form a rainbow $2K_2$. ■

Now we will show the exact values of $rb(\mathcal{T}_n, 3K_2)$ for all $n \geq 6$. In the first place, we give the exact value of $rb(\mathcal{T}_n, 3K_2)$ for $n = 6$.