Discrete Mathematics 331 (2014) 165–174

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Cut equivalence of d-dimensional guillotine partitions☆

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ARTICLE INFO

Article history:
Received 1 December 2012
Received in revised form 16 May 2014
Accepted 17 May 2014
Available online 2 June 2014

Keywords:
Guillotine partitions
Generating functions
Inclusion–exclusion principle

ABSTRACT

A *guillotine partition* of a d-dimensional axis-aligned box B is a recursive partition of B by axis-aligned hyperplane cuts. The size of a guillotine partition is the number of boxes it contains. Two guillotine partitions are *box-equivalent* if their boxes satisfy compatible order relations with respect to the axes. (In many works, box-equivalent guillotine partitions are considered identical.) In the present work we define *cut-equivalence* of guillotine partitions, derived in a similar way from order relations of cuts. We prove structural properties related to these kinds of equivalence, and enumerate cut-equivalence classes of d-dimensional guillotine partitions of size n.

1. Introduction

1.1. Basic definitions

Let B be a d-dimensional axis-aligned box. A *guillotine partition* of B is either the trivial partition (whose only part is B itself) or a partition obtained by cutting B by a hyperplane which is perpendicular to an axis x_i, $1 \le i \le d$, into two sub-boxes whose inner partitions are also guillotine (in a recursive way). The *size* of a guillotine partition is the number of (unpartitioned) boxes in it. We often denote by B the partition as well as the box. Fig. 1 shows two 3-dimensional guillotine partitions of size 6.

Guillotine partitions have been studied intensively due to their important role in geometric algorithms, visualization of scientific data, integrated circuit design, and many more fields. Stockmeyer [10], Du et al. [5], Gonzalez and Zheng [7] suggested algorithms for approximating the minimum edge-length guillotine partition in two dimensions. Mitchell [8] and Cardei et al. [4] developed polynomial-time approximation schemes (PTASs) for this problem.

Yao et al. [11] were the first to show that the number of combinatorial types (or, in our terms, B-equivalence classes) of planar guillotine partitions of size $n+1$ is the nth Schröder number. This was generalized to higher-dimensional guillotine partitions by Ackerman et al. [1]; their representation by so-called "separable multidimensional permutations" was suggested by Asinowski and Mansour [3]. There are plenty more works on computational, optimization, and approximation aspects of guillotine partitions.

☆ Work on this paper by the first and second authors has been supported in part by a grant from Bar-Nir Bergreen Software Technology Center of Excellence.

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http://dx.doi.org/10.1016/j.disc.2014.05.014
0012-365X/

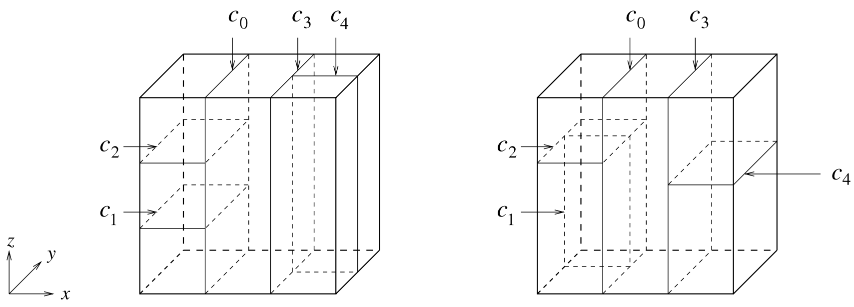

Fig. 1. Two (C-equivalent) guillotine partitions of a 3-dimensional box.

Understanding the combinatorial structure of guillotine partitions is, therefore, important not only from the combinatorial point of view, but also for analyzing the efficiency of data structures that hold the partitions, and the running times of algorithms that generate them. In many works, guillotine partitions that have the same recursive structure with respect to their boxes are considered identical. However, another kind of elements in guillotine partitions is their cuts. In some applications, the structure of the cuts is more relevant to the complexity or running-time analysis than the structure of the boxes. For example, suppose that we fix a point set P of size n (being in general position in the sense that no two points of P belong to the same axis-aligned hyperplane) in a d-dimensional box B, and consider guillotine partitions of B such that each point of P belongs to exactly one cut. The combinatorial description of cut-point incidence in this case involves rather cuts than boxes, and the "cut-equivalence" is a natural way to identify guillotine partitions. Alternatively, one can be interested in arrangements of axis-aligned $(d-1)$-dimensional boxes (e.g., rectangles in the 3-space), such that the natural neighborhood relations between them are important, but the exact orientation (that is, being orthogonal to a specific axis) is not essential and can be chosen according to some additional parameters. If such an arrangement induces a guillotine partition, then our notion of cut-equivalence captures properly its combinatorial structure. In addition, this equivalence allows to define a subclass of guillotine partitions with a simplified structure—without so-called "improper pairs" (see Section 4).

Thus, the goal of this paper is to study systematically these two types of structures. To this aim, we define two kinds of equivalence of d-dimensional guillotine partitions, namely, B-equivalence and C-equivalence,[2] in terms of order relations between boxes and cuts, respectively. We demonstrate that B-equivalence is in fact the "usual" way to identify guillotine partitions, while C-equivalence is a coarser way to do it. We also show how C-equivalence is related to B-equivalence (Propositions 17 and 18), and use this result to obtain an enumeration of C-equivalence classes (Theorem 2). An important issue here is the asymptotic enumeration of cuts. How much do we save if we create all possible C-equivalence classes (or, equivalently, all B-equivalence classes without "improper pairs") instead of B-equivalence classes? We show that for C-equivalence, the asymptotic growth rate is roughly one half of that for B-equivalence.

The intersection of a box B with a hyperplane that splits it into two sub-boxes is a *primary cut* (for example, c_0 and c_3 in Fig. 1 are primary cuts). If either of these boxes is further partitioned, we can speak about its primary cut as well. A *cut* in a guillotine partition is either a primary cut of the whole box, or (in a recursive manner) a (primary) cut in the partition of one of the sub-boxes. It is assumed that parallel cuts do not intersect, that is, they cannot share a $(d-2)$-dimensional "edge". It is easy to see by induction that a guillotine partition B of size $n+1$ (which will be denoted by $|B| = n+1$) has exactly n cuts.

Throughout this paper, the dimension d is assumed fixed, and all the guillotine partitions are assumed to be d-dimensional.

If a nontrivial guillotine partition B has several primary cuts, then they are all perpendicular to the same axis. If the primary cut(s) of a nontrivial guillotine partition B is (are) perpendicular to the x_i axis, we say that B is *x_i-aligned*. The parts of B bounded by two consecutive primary cuts, as well as the part below the lowest (with respect to x_i) primary cut, and the part above the highest primary cut, will be called *slices* and denoted by $S_1, \ldots, S_k$ (ordered from bottom to top with respect to x_i). A *trivial slice* is a slice of size 1. A *2-slice* is a slice of size 2. Primary cuts of any slice are not parallel to those of B; that is, any nontrivial slice is aligned differently from B. The guillotine partition in Fig. 1 is x-aligned, and it has three slices: S_1 is a z-aligned slice of size 3, S_2 is a trivial slice, and S_3 is a y-aligned 2-slice. The lowest primary cut with respect to x_i (where x_i is as above) is called the *principal cut* of B. The sub-boxes obtained by cutting B along the principal cut are denoted by B^- (the part of B below the principal cut, that is, the lowest slice) and B^+ (the part of B above the principal cut). In Fig. 1, the principal cut of B is c_0, the principal cut of B^- is c_1, and the principal cut of B^+ is c_3.

1.2. Order relations in guillotine partitions

We define d order relations between boxes and between cuts in d-dimensional guillotine partitions.

Definition 1. Consider a nontrivial d-dimensional guillotine partition B with principal cut c.

[2] B stands for *boxes*, C for *cuts*.