Discrete Mathematics 331 (2014) 175–187

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Some new results on mutually orthogonal frequency squares[☆]

Mingchao Li [a,b], Yan Zhang [a], Beiliang Du [a,*]

[a] Department of Mathematics, Soochow University (Suzhou University), Suzhou 215006, PR China
[b] College of Science, Hebei University of Engineering, Handan 056038, PR China

ARTICLE INFO

Article history:
Received 6 January 2014
Received in revised form 8 May 2014
Accepted 9 May 2014
Available online 3 June 2014

Keywords:
Frequency square
Transversal design
Difference matrix

ABSTRACT

Let $n = \lambda m$. A frequency square $F(n; \lambda)$ is an $n \times n$ array in which each of m distinct symbols appears exactly λ times in each row and column. Two such squares are said to be orthogonal if upon superposition, each of the m^2 distinct ordered pairs occurs exactly λ^2 times. Hedayat, Sloane and Stufken in their book Hedayat et al. (1999) provided a small table of lower bounds for the maximum number of mutually orthogonal frequency squares of type $F(n; \lambda)$ (HSS Table), and posed the following research problem: Improve the lower bound for the entries in HSS Table.

Laywine and Mullen in 2001 extended the table (LM Table). In this article we will give some new construction methods of mutually orthogonal frequency squares. As an application we improve the lower bounds for more than half of the entries in LM Table.

1. Introduction

A *frequency square* $F(n; \lambda_1, \lambda_2, \ldots, \lambda_m)$ is an $n \times n$ array consisting of the number $1, 2, \ldots, m$ with the property that for each $i = 1, 2, \ldots, m$, the number i occurs exactly λ_i times in each row and each column. Clearly $n = \lambda_1 + \lambda_2 + \cdots + \lambda_m$ and an $F(n; 1, \ldots, 1)$ frequency square is a Latin square of order n. If all of the λ_i's are equal to λ, we will simply write $F(n; \lambda)$. Two frequency squares $F_1(n, \lambda_1, \lambda_2, \ldots, \lambda_{m_1})$ and $F_2(n; \mu_1, \mu_2, \ldots, \mu_{m_2})$ are said to be *orthogonal* if upon superposition, each ordered pair (i, j) occurs exactly $\lambda_i\mu_j$ times for $i = 1, 2, \ldots, m_1, j = 1, 2, \ldots, m_2$. A set $\{F_1, F_2, \ldots, F_t\}$ of $t \geq 2$ frequency squares is said to be orthogonal if F_i is orthogonal to F_j whenever $i \neq j$.

Mutually orthogonal frequency squares (MOFS) have been studied by some researchers and found to have a number of applications. In statistics they are primarily used in designing experiments, Hedayat, Sloane and Stufken in [13] pointed out: Although Latin squares have many useful properties, for some statistical applications these structures are too restrictive. The more general concepts of frequency squares and orthogonal frequency squares offer more flexibility. Anthony, Martin, Seberry and Wild in [5] gave an application in cryptography.

We use the notation t $MOFS(n; \lambda)$ to denote t mutually orthogonal frequency squares of type $F(n; \lambda)$, and the notation $f(n; \lambda)$ to denote the lower bounds for the maximum number of mutually orthogonal frequency squares of type $F(n; \lambda)$, that is, the number of MOFS that can currently be constructed of type $F(n; \lambda)$. It is easy to see that $F(n; 1)$ is a Latin square of order n, $tMOFS(n; 1)$ are t mutually orthogonal Latin squares of order n, $tMOLS(n)$. $f(n; 1)$ is the lower bounds for the maximum number of mutually orthogonal Latin squares of order n. From Hedayat, Raghavarao and Seiden [12] we know

$$f(n, \lambda) \leq \frac{(n-1)^2}{m-1}.$$

☆ This work was supported by the National Natural Science Foundation of China (Grant Nos. 11171248, 11301370).
* Corresponding author.
E-mail address: dubl@suda.edu.cn (B. Du).

http://dx.doi.org/10.1016/j.disc.2014.05.006

Hedayat, Sloane and Stufken in their book [13] provided a small table of the lower bounds for the maximum number of MOFS of type $F(n;\lambda)$ (HSS Table), and posed the following research problem (Research Problem 8.22 in [13]).

Research problem Improve the lower bounds for the entries in HSS Table and extend the table.

In 2001, Laywine and Mullen [15] extended HSS Table (LM Table). But for $(n, \lambda) = (44, 22)$ and $(88, 22)$ in LM Table, $f(n;\lambda)$ should be 1849 and 86 respectively as constructed by Laywine and Mullen in [15]. Moreover from Finney [10], we know that $f(6; 2) = 7$ and $f(6; 3) = 8$. In LM Table, they were written as $f(6; 2) = 8$ and $f(6; 3) = 7$. Applying them to the constructions in [15] yields some entries of $f(n;\lambda)$ not the same as those in LM Table. We show them in Table 1.1 in bold.

In this paper we will give some construction methods of MOFS. As an application we improve the lower bounds for more than half of entries in LM Table. We show them in Table 1.1, where we write $f(n;\lambda)$ simply as f, f_{LM} denotes the value of $f(n,\lambda)$ in LM Table (with the corrections mentioned above) and f_{New} denotes the value of $f(n,\lambda)$ in our article.

2. Preliminaries

In this section we introduce some of the auxiliary designs and establish some of the fundamental results which will be used later. A *transversal design* of group size n, block size k and index λ, denoted by $TD_\lambda(k, n)$, is a triple $(X, \mathcal{G}, \mathcal{B})$ where: X is a set of kn elements; $\mathcal{G}$ is a partition of X into k classes (called *groups*), each of size n; $\mathcal{B}$ is a collection of k-subsets of X (called *blocks*); and every unordered pair of elements from X is either contained in exactly one group, or contained in exactly λ blocks, but not both. When $\lambda = 1$, one writes simply $TD(k, n)$. Greig and Colbourn in [11] gave a table of $TD_\lambda(k, n)$ with $1 \le \lambda \le 36$ and $2 \le n \le 50$. It is well known that the existence of a $TD(k, n)$ is equivalent to the existence of $k-2$ $MOLS(n)$, that is, $k-2$ $MOFS(n; 1)$, and the existence of a $TD_\lambda(k, n)$ is equivalent to the existence of an $(n, k; \lambda)$-net (see, for example, [6]). An $(n, k; \lambda)$-net is a set X of λn^2 elements together with a set $\mathcal{D}$ of kn subsets of X (called blocks) each of size λn. The set of all blocks is partitioned into k parallel classes, each containing n disjoint blocks. Every two nonparallel blocks intersect in λ elements.

Lemma 2.1. *If there exist a $TD_{\frac{\lambda}{m}}(r, m)$ and t $MOLS(\lambda m)$, then there exist $s + (t - s)r$ MOFS of order λm, which consist of s $MOLS(\lambda m)$ and $(t - s)r$ $MOFS(\lambda m; \lambda)$, for any non-negative integer $s < t$.*

Proof. Notice that the existence of a $TD_{\frac{\lambda}{m}}(r, m)$ is equivalent to the existence of an $(m, r; \frac{\lambda}{m})$-net, let $(X, \mathcal{D})$ be the $(m, r; \frac{\lambda}{m})$-net and $\Gamma = \{C_1, C_2, \ldots, C_r\}$ be all the parallel classes in X, where $C_i = \{B_{i1}, B_{i2}, \ldots, B_{im}\}$, $i = 1, 2, \ldots, r$. Let $\{L_1, \ldots, L_s, L_{s+1}, \ldots, L_t\}$ be the set of MOLS of order λm. We may assume the λm distinct symbols in the Latin squares are just the λm points in X, say $1, 2, \ldots, \lambda m$. To any $C_i \in \Gamma$ and any $L_j \in \{L_{s+1}, \ldots, L_t\}$, we define a $\lambda m \times \lambda m$ square F_{ij} from L_j by mapping $\{1, 2, \ldots, \lambda m\}$ onto $\{1, 2, \ldots, m\}$ in such a way that

$$\sigma_i(x) = h, \quad x \in B_{ih},\ 1 \le h \le m.$$

It is obvious that F_{ij} is a frequency square of type $F(\lambda m; \lambda)$.

We now show two distinct frequency squares defined above are orthogonal.

First consider the orthogonality of $F_{i_1j_1}$ and $F_{i_2j_2}$ where $j_1 \neq j_2$. Symbol α in $F_{i_1j_1}$ and symbol β in $F_{i_2j_2}$ are derived from $B_{i_1h_1}$ in C_{i_1} and $B_{i_2h_2}$ in C_{i_2} respectively. To any $\alpha' \in B_{i_1h_1}$ and any $\beta' \in B_{i_2h_2}$, the pair (α', β') occurs exactly once when L_{j_1} is superimposed on L_{j_2} since L_{j_1} and L_{j_2} are orthogonal. So the pair (α, β) occurs exactly $\lambda \cdot \lambda = \lambda^2$ times when $F_{i_1j_1}$ is superimposed on $F_{i_2j_2}$.

Next we consider the orthogonality of $F_{i_1j_1}$ and $F_{i_2j_2}$ where $j_1 = j_2$. Then $i_1 \neq i_2$. Symbol α in $F_{i_1j_1}$ and symbol β in $F_{i_2j_2}$ are derived from $B_{i_1h_1}$ in C_{i_1} and $B_{i_2h_2}$ in C_{i_2} respectively. Since C_{i_1} and C_{i_2} are two parallel classes of X, we have $|B_{i_1h_1} \cap B_{i_2h_2}| = \frac{\lambda}{m}$. So the pair (α, β) occurs exactly $\frac{\lambda}{m}$ times in each of the λm rows if $F_{i_1j_1}$ is superimposed on $F_{i_2j_2}$. Since $\frac{\lambda}{m} \cdot \lambda m = \lambda^2$, the orthogonality condition is satisfied.

Now there are r possible parallel classes C_i and $t - s$ possible Latin squares L_j. So we obtain $(t - s)r$ $MOFS(\lambda m; \lambda)$.

Finally we show L is orthogonal to F_{ij} defined above, where $L \in \{L_1, \ldots, L_s\}$.

Symbol β in F_{ij} is derived from B_{ih} in C_i. The pair consisting of symbol α in L and any β' of B_{ih} occurs exactly once when L is superimposed on L_j. So the pair (α, β) occurs exactly λ times if L is superimposed on F_{ij}.

Therefore we obtain the result. □

When $s = 0$ and 1, we have the following corollary.

Corollary 2.2. 1. *If there exist a $TD_{\frac{\lambda}{m}}(r, m)$ and t $MOLS(\lambda m)$, then there exist $1 + (t - 1)r$ MOFS of order λm, which consist of a Latin square of order λm and $(t - 1)r$ $MOFS(\lambda m; \lambda)$.*

2. *If there exist a $TD_{\frac{\lambda}{m}}(r, m)$ and t $MOLS(\lambda m)$, then there exist tr $MOFS(\lambda m; \lambda)$.*

Lemma 2.3. *There exist 6 MOFS of order 4, which consist of a Latin square of order 4 and 6 $MOFS(4; 2)$.*

Proof. It comes from Corollary 2.2 with $k = m = 2$ and $r = 3$. The conditions $TD(3, 2)$ and 3 $MOLS(4)$ we need come from [6]. □