Discrete Mathematics 327 (2014) 1–8

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The edge-face choosability of plane graphs with maximum degree at least 9

Xiaoxue Hu [a], Weifan Wang [a,*], Yiqiao Wang [b]

[a] *Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China*
[b] *School of Management, Beijing University of Chinese Medicine, Beijing 100029, China*

ARTICLE INFO

Article history:
Received 2 September 2013
Received in revised form 11 March 2014
Accepted 14 March 2014
Available online 28 March 2014

Keywords:
Plane graph
Coloring
Edge-face choosability
Maximum degree

ABSTRACT

A plane graph G is said to be edge-face k-choosable if, for every list L of colors satisfying $|L(x)| = k$ for $x \in E(G) \cup F(G)$, there exists a coloring which assigns to each edge and face a color from its list so that any adjacent or incident elements receive different colors. In this paper, we prove that every plane graph G with maximum degree $\Delta(G) \geq 9$ is edge-face $(\Delta(G) + 1)$-choosable.

© 2014 Elsevier B.V. All rights reserved.

1. Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges unless otherwise stated. A *plane graph* is a particular drawing in the Euclidean plane of a planar graph. For a plane graph G, we denote its vertex set, edge set, face set, maximum degree, and minimum degree by $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$ and $\delta(G)$, respectively.

A plane graph G is *edge-face k-colorable* if $E(G) \cup F(G)$ can be colored with k colors such that any two adjacent or incident elements receive different colors. The *edge-face chromatic number* $\chi_{ef}(G)$ of G is defined to be the least integer k such that G is edge-face k-colorable.

A mapping L is said to be an assignment for the plane graph G if it assigns a list $L(x)$ of possible colors to each element x in $E(G) \cup F(G)$. If G has an edge-face coloring ϕ such that $\phi(x) \in L(x)$ for all elements x, then we say that G is *edge-face L-colorable* or ϕ is an edge-face L-coloring of G. G is edge-face k-choosable if it is edge-face L-colorable for every list assignment L satisfying $|L(x)| = k$ for all elements $x \in E(G) \cup F(G)$. The *list edge-face chromatic number* $\chi^L_{ef}(G)$ of G is the smallest integer k such that G is edge-face k-choosable. By definition, it is straightforward to derive that $\chi^L_{ef}(G) \geq \chi_{ef}(G) \geq \Delta(G)$.

The edge-face colorings of plane graphs were first studied by Jucovič [5] and Fiamčík [3], who investigated the case of 3- and 4-regular plane graphs. In 1975, Mel'nikov [9] conjectured that every plane graph G is edge-face $(\Delta(G)+3)$-colorable. Wang [15], and independently Hu et al. [4] confirmed the conjecture for the case $\Delta(G) \leq 3$. Wang and Zhang [18] further settled the case $\Delta(G) = 4$. Two similar, yet independent, proofs of this conjecture were given by Waller [14] and Sanders and Zhao [10]. Both proofs made use of the Four-Color Theorem. Without applying the Four-Color Theorem, Wang and Lih [16], and independently, Sanders and Zhao [12], gave a new proof of this conjecture.

Borodin [1] proved that every plane graph G with $\Delta(G) \geq 10$ is edge-face $(\Delta(G) + 1)$-colorable. Recently, the condition that $\Delta(G) \geq 10$ is reduced to $\Delta(G) = 9$ by Sereni and Stehlík [13], and independently, by Macon [8], and further to $\Delta(G) = 8$

* Corresponding author.
E-mail address: wwf@zjnu.cn (W. Wang).

http://dx.doi.org/10.1016/j.disc.2014.03.008
0012-365X/© 2014 Elsevier B.V. All rights reserved.

by Kang et al. [6]. These results imply that every plane graph G with $\Delta(G) \geq 7$ is edge-face $(\Delta(G)+2)$-colorable. Sanders and Zhao [11] proved that every plane graph G with $\Delta(G) = 3$ is edge-face 5-colorable. More recently, Chen, Raspaud and Wang [2] proved that every plane graph G with $\Delta(G) = 6$ is edge-face 8-colorable.

In [17], Wang and Lih investigated the list edge-face coloring of plane graphs. They proved that every plane graph G is edge-face $(\Delta(G)+3)$-choosable. This is a reinforcement of the results in [10,12,14,16]. In this paper, we consider the edge-face choosability of plane graph with large maximum degree. More precisely, we show the following:

Theorem 1. *If G is a plane graph with $\Delta(G) \geq 9$, then $\chi^L_{ef}(G) \leq \Delta(G)+1$.*

Theorem 1 extends and improves the result of [1,13]. The organization of this paper is as follows. In Section 2, we collect some notation and basic definitions used in the subsequent sections. In Section 3, we establish a structural lemma, which plays a key role in the proof of the main result. The proof of Theorem 1 is postponed to Section 4. Some open problems are proposed in Section 5.

2. Notation

Let G be a plane graph with $\delta(G) \geq 2$. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1u_2\cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices of $b(f)$ in clockwise order. Repeated occurrences of a vertex are allowed. The degree of a face is the number of edge-steps in its boundary walk. Note that each cut-edge is counted twice. For $x \in V(G) \cup F(G)$, let $d_G(x)$ denote the degree of x in G. A vertex of degree k (at most k, at least k, respectively) is called a k-vertex (k^--vertex, k^+-vertex, respectively). Similarly, we can define k-face, k^--face and k^+-face. When v is a k-vertex, we say that there are k faces incident to v. However, these faces are not required to be distinct, i.e., v may have repeated occurrences on the boundary walk of some of its incident faces. Sometimes, v is said to be a $(a_1, a_2, \ldots, a_k)$*-vertex* if it is incident to k faces $f_1, f_2, \ldots, f_k$ in clockwise order with $d_G(f_i) = a_i$ for $i = 1, 2, \ldots, k$. However, some of these faces may be identical.

For a face $f \in F(G)$, let $m(f)$ denote the number of different faces incident to f and $m_{4^-}(f)$ denote the number of different 4^--faces incident to f, respectively. We say that f is *light* if f is adjacent to a 4^--face f', $d_G(f) + m(f) - m_{4^-}(f) \leq 9$, and $b(f) \cap b(f')$ contains a 2-vertex.

For an edge $e = xy \in E(G)$, let $t(e)$ and $q(e)$ denote the number of 3-faces and 4-faces incident to e, respectively. Let $l(e) = t(e) + q(e)$. If $l(e) \geq 1$ and $d_G(x) + d_G(y) - l(e) \leq 9$, then e is called a *light edge*. We say that e is *triangular* if $t(e) \geq 1$, and *fully-triangular* if $t(e) = 2$. Note that a fully-triangular edge is also a triangular edge, but not vice versa. A vertex v is *triangular* (or *fully-triangular*) if each edge incident to v is triangular (or fully-triangular).

A cycle C of the plane graph G is called *separating* if both its interior and exterior contain at least one vertex of G. Let $V^0(C)$ denote the set of vertices in G that lie interior to C. A 2-vertex is called *bad* if it lies in a separating 3-cycle. Otherwise, it is called *good*.

3. A structural lemma

Lemma 2. *Let G be a connected plane graph with $\delta(G) \geq 2$. Then G contains one of the following configurations:*

(C1) *A 2-vertex incident to a 4^--face and a 5^--face.*
(C2) *A 4-cycle $v_1v_2v_3v_4v_1$ satisfying that for $i \in \{1, 3\}$, v_i is either a triangular 2-vertex or a $(3, 4^-, 4^-)$-vertex.*
(C3) *A good 2-vertex adjacent to a 5^--vertex.*
(C4) *A 5-face incident to an edge uv with $d_G(u) + d_G(v) \leq 9$.*
(C5) *A light edge.*
(C6) *A light face.*
(C7) *A face $f = [u_1u_2\cdots u_{2s}]$ with $s \geq 2$ such that for $i = 1, 3, \ldots, 2s-1$, $d_G(u_i) = 2$ and u_i is incident to a 4^--face.*

Proof. Assume to the contrary that the lemma is false and G is a counterexample. Then G is a connected plane graph with $\delta(G) \geq 2$ and containing none of the configurations (C1)–(C7). To obtain a contradiction by discharging analysis, we need to define a new graph H from G in the following way: If there are no bad 2-vertices in G, we put $H = G$. Otherwise, choose a separating 3-cycle T with the least interior vertices and passing through a bad 2-vertex, and put $H = G[V^0(T) \cup V(T)]$. In the following, we call vertices in $V^0(T)$ *internal* vertices of H and vertices in $V(T)$ T-vertices of H (if T exists). For each $v \in V^0(T)$, it is evident that $d_H(v) = d_G(v)$. Since G contains no (C3), there are no two adjacent internal 2-vertices in H.

Because G contains no (C1) and (C5), the following Claim 1 holds automatically.

Claim 1. *Let x and y be two adjacent internal vertices with $l(xy) \geq 1$.*

(1) *If $d_H(x) = 2$, then $d_H(y) \geq 9$.*
(2) *Assume that $d_H(x) = 3$. If $l(xy) = 2$, then $d_H(y) \geq 9$; If $l(xy) = 1$, then $d_H(y) \geq 8$.*
(3) *Assume that $d_H(x) = 4$. If $l(xy) = 2$, then $d_H(y) \geq 8$; If $l(xy) = 1$, then $d_H(y) \geq 7$.*
(4) *Assume that $d_H(x) = 5$. If $l(xy) = 2$, then $d_H(y) \geq 7$; If $l(xy) = 1$, then $d_H(y) \geq 6$.*

Download English Version:

https://daneshyari.com/en/article/4647272

Download Persian Version:

https://daneshyari.com/article/4647272

Daneshyari.com