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Planar graphs with cycles of length neither 4 nor 7 are (3, 0, 0)-colorable☆

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ABSTRACT

Let $d_1, d_2, \ldots, d_k$ be k non-negative integers. A graph G is $(d_1, d_2, \ldots, d_k)$-colorable if the vertex set of G can be partitioned into subsets $V_1, V_2, \ldots, V_k$ such that the subgraph $G[V_i]$ induced by V_i has maximum degree at most d_i for $1 \le i \le k$. It is known that planar graphs with cycles of length neither 4 nor k, $k \in \{5, 6\}$, are (3, 0, 0)-colorable. In this paper, we show that planar graphs with cycles of length neither 4 nor 7 are also (3, 0, 0)-colorable.

1. Introduction

Let $d_1, d_2, \ldots, d_k$ be k non-negative integers. A graph G is called *improperly* $(d_1, d_2, \ldots, d_k)$*-colorable*, or simply, $(d_1, d_2, \ldots, d_k)$*-colorable*, if we can use colors from $\{1, 2, \ldots, k\}$ to color the vertices of G such that $G[V_i]$, the subgraph of G induced by V_i, has maximum degree at most d_i, where V_i is the subset of vertices colored i for every $i \in \{1, 2, \ldots, k\}$. With this notion, the well-known Four Color Theorem [1,2] says that every planar graph is (0, 0, 0, 0)-colorable; the well-known Three Color Theorem of Grötzsch [5] says that every triangle-free planar graph is (0, 0, 0)-colorable. How about if we color C_i-free planar graphs with only three colors for $i \ge 4$? Steinberg conjectured [8] that every planar graph with cycles of length neither 4 nor 5 is also (0, 0, 0)-colorable. Motivated by Steinberg's conjecture, Lih et al. [7] proved that every planar graph with cycles of length neither 4 nor l for some $l \in \{5, 6, 7\}$ is (list) (1, 1, 1)-colorable. Dong and Xu [4] extended this to $l \in \{8, 9\}$. Motivated by Bordeaux conjecture (stronger than Steinberg's), Xu [9] proved that every planar graph with neither adjacent triangles nor cycle of length 5 is (1, 1, 1)-colorable. Recently, motivated by Steinberg's conjecture again, Chang et al. [3] proved that every planar graph with cycles of length neither 4 nor 5 is (4, 0, 0)- and (2, 1, 0)-colorable. Very recently, Hill et al. [6] showed that every planar graph with cycles of length neither 4 nor 5 is (3, 0, 0)-colorable; Xu and Wang [10] showed that every planar graph with cycles of length neither 4 nor 6 is (3, 0, 0)-colorable. In this paper, we show

Theorem 1. *Every planar graph with cycles of length neither* 4 *nor* 7 *is* (3, 0, 0)*-colorable.*

The rest of this section is devoted to some terminology and notation used later. All graphs considered in this paper are finite, simple and undirected. Call a graph G *planar* if it can be embedded into the plane so that its edges meet only at their ends. Any such a particular embedding of a planar graph is called a *plane* graph. For a plane graph G, we use V, E, F and δ to denote its vertex set, edge set, face set and minimum degree, respectively. For a vertex $v \in V$, let $d(v)$ denote the *degree* of

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v in G, i.e., the number of edges incident with v in G. Call v a *k-vertex*, a *k^+-vertex*, or a *k^--vertex* if $d(v) = k$, $d(v) \geq k$, or $d(v) \leq k$, respectively. An edge $xy \in E$ is called a *$(d(x), d(y))$-edge*, and x is called a *$d(x)$-neighbor* of y. For a face $f \in F$, the length of the boundary of f, denoted $d(f)$, is called the *degree* of f. Call f a *k-face*, a *k^+-face*, or a *k^--face* if $d(f) = k$, $d(f) \geq k$, or $d(f) \leq k$, respectively. We write $f = [v_1v_2 \ldots v_k]$ if $v_1, v_2, \ldots, v_k$ are consecutive vertices on f in a cyclic order, and we say that f is a *$(d(v_1), d(v_2), \ldots, d(v_k))$-face*. A *k-cycle* is a cycle of length k. Here, a *triangle* is a 3-face. Call a vertex or an edge *triangular* if it is incident with a triangle; *non-triangular* otherwise. Call u a *triangular neighbor* of v if uv is a triangular edge. Let k be a positive integer. Call a vertex *vk-triangular* if it is incident with k non-adjacent triangles. In this paper, the non-adjacency of two triangles T_1 and T_2 incident with a vertex v in G is ensured by the fact that G has no 4-cycles. For the same reason, besides the edges of T_1 and T_2, no edge in G connects T_1 and T_2. Let $T = [uvw]$ be a triangle of G. Call u, as well as w, a *non-isolated* neighbor of v, and u and w a *couple* of *non-isolated* neighbors of v. Call a neighbor v' of v *isolated* if no triangle in G contains vv'. Furthermore, v' is called an *isolated k-neighbor* of v if $d(v') = k$; and v is called a *pendent neighbor of* v', or $T = [uvw]$ is called a *pendent 3-face* of v' if $d(v) = 3$.

2. Reducible configurations

As usual, to *properly* color a vertex v means to assign v a color which has not been assigned to any neighbor of v. In this paper, *to* $(3, 0, 0)$*-color*, in short, *to color*, a vertex v, is to properly color v with 2 or 3, or color v with 1 when color 1 is used on the neighbors of v at most three times. Let G' be a subgraph of G, and φ a $(3, 0, 0)$-coloring of G'. Let $A \subset V(G')$, we define $\varphi(A) = \{\varphi(a)|a \in A\}$. Note that $\varphi(A)$ may be a multi-set of colors. Let v be a vertex of G. For convenience, we use $Im_\varphi(v, 1)$ to denote the number of neighbors of v being colored 1.

Suppose Theorem 1 is false. Let $G = (V, E)$ be a counterexample to Theorem 1 with the fewest vertices. Clearly G is connected. Below are some structural properties of G.

Lemma 2.1 ([6]). $\delta(G) \geq 3$. □

Lemma 2.2 ([6]). *Every* 3*-vertex in G has at least one* 6^+*-neighbor.* □

Let $f = [uvw]$ be a 3-face. Call f *poor* if $d(u) = d(v) = 3$ and both the isolated neighbors of u and v have degrees at most 5; *semi-poor* if $d(u) = d(v) = 3$ and exactly one of the isolated neighbors of u and v has degree at most 5; *rich* otherwise. A *poor 3-vertex* is a 3-vertex that is on a poor or a semi-poor 3-face and has an isolated 5^--neighbor. By definition and Lemma 2.2, a poor 3-face has exactly two poor 3-vertices, a semi-poor 3-face has exactly one poor 3-vertex, and a rich 3-face has no poor 3-vertex.

Lemma 2.3 ([6]). *Every* $(3, 3, 6^-)$*-face in G is rich.*

Lemma 2.4. *If* $f = [uvw]$ *is a* $(3, 3, 4)$*-face with* $d(u) = d(v) = 3$ *and* $d(w) = 4$*, then* w *has at least one* 6^+*-neighbor.*

Proof. Let w_1 and w_2 be the two neighbors of w not on f. Suppose both w_1 and w_2 are 5^--vertices. Without loss of generality, we may assume that $d(w_1) = d(w_2) = 5$. By the minimality of G, the graph $G - v$ admits a $(3, 0, 0)$-coloring φ. Clearly, the three neighbors of v are colored distinct under φ, since otherwise v could be properly colored. If $\varphi(u) = 1$ or $\varphi(w) = 1$, then we could color v with 1 since at that time u or w has at most two neighbors colored 1. Thus the isolated neighbor of v is colored 1. By the symmetry of the colors 2 and 3, we may assume that $\varphi(w) = 2$ and $\varphi(u) = 3$.

Under this situation, at least one of w_1 and w_2 is colored 1, since otherwise, we could recolor w with 1, then color v with 2. Observe that if w_i is colored 1 and $Im(w_i, 1) = 3$, then we could recolor w_i with 2 or 3. So we may assume that, for $i = 1, 2$, $Im(w_i, 1) \leq 2$ if w_i is colored 1. Now we can recolor w with 1 and then color v with 2. Giving a $(3, 0, 0)$-coloring of G, a contradiction. □

Lemma 2.5. *Let* v *be a* 2*-triangular* 7*-vertex with seven* 3*-neighbors. If one of the two triangles incident with* v *is poor or semi-poor, then the other is rich.*

Proof. Let $v_1, v_2, \ldots, v_7$ be the seven 3-neighbors of v, where $T_1 = [vv_1v_2]$, $T_2 = [vv_3v_4]$ are the two triangles defining v to be 2-triangular. Assume that T_1 is poor or semi-poor with v_2 poor. Suppose T_2 is not rich with v_4 poor. For $i = 1, 2, 3, 4$, let v_i' be the isolated neighbor of v_i. By the definition of a poor vertex, $d(v_i') \leq 5$ for $i = 2, 4$. By the minimality of G, the graph $G - \{v, v_1, v_2, \ldots, v_7\}$ admits a $(3, 0, 0)$-coloring φ. We first extend φ to $G - v$ in such way: for $i = 1, 3$, we properly color v_i with a color in $\{2, 3\} \setminus \{\varphi(v_i')\}$. Then we properly color v_j for $j = 2, 4, 5, 6, 7$. We are going to show that we can get a $(3, 0, 0)$-coloring of G, a contradiction proving the lemma.

Since $\varphi(v_1), \varphi(v_3) \in \{2, 3\}$, $Im_\varphi(v, 1) \leq 5$. On the other hand, $Im_\varphi(v, 1) \geq 4$, since otherwise v could be colored with 1. First suppose $Im_\varphi(v, 1) = 5$. If $\varphi(v_1) = \varphi(v_3)$, then we could color v with $\{2, 3\} \setminus \{\varphi(v_1)\}$. Otherwise we could recolor v_1 with a color in $\{1, \varphi(v_3)\} \setminus \{\varphi(v_1')\}$, and then color v with $\varphi(v_1)$. Next suppose $Im_\varphi(v, 1) = 4$. By the symmetry of the colors 2 and 3, we may assume that $\varphi(N(v)) = \{1, 1, 1, 1, 2, 2, 3\}$. According to which vertex colored 3, there are three cases under consideration.