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Rainbow connection for some families of hypergraphs

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ABSTRACT

An edge-coloured path in a graph is *rainbow* if its edges have distinct colours. The *rainbow connection number* of a connected graph G, denoted by $rc(G)$, is the minimum number of colours required to colour the edges of G so that any two vertices of G are connected by a rainbow path. The function $rc(G)$ was first introduced by Chartrand et al. (2008), and has since attracted considerable interest. In this paper, we introduce two extensions of the rainbow connection number to hypergraphs. We study these two extensions of the rainbow connection number in minimally connected hypergraphs, hypergraph cycles and complete multipartite hypergraphs.

1. Introduction

In this paper, we shall consider hypergraphs which are finite, undirected and without multiple edges. For any undefined terms we refer to [1]. Also, for basic terminology for graphs we refer to [2].

The concept of *rainbow connection* in graphs was first introduced by Chartrand et al. [5] in 2008. An edge-coloured path is *rainbow* if the colours of its edges are distinct. For a connected graph G, the *rainbow connection number* of G, denoted by $rc(G)$, is the minimum integer t for which there exists a colouring of the edges of G with t colours such that, any two vertices of G are connected by a rainbow path. In their original paper, Chartrand et al. [5] studied the function $rc(G)$ for many graphs G, including when G is a tree, a cycle, a wheel, and a complete multipartite graph. Since then, the rainbow connection subject has attracted considerable interest. Many results about $rc(G)$ have been proved when G satisfies some property, such as a minimum degree condition, a diameter condition, a connectivity condition, and when G is a regular graph or a random graph. Several related functions have also been introduced and studied. These include the *rainbow k-connection number* $rc_k(G)$ and the *rainbow vertex connection number* $rvc(G)$. See for example, Caro et al. [3], Chandran et al. [4], Chartrand et al. [6], Fujita et al. [7], Krivelevich and Yuster [9], and Li et al. [10], among others. A survey by Li et al. [11] and a book by Li and Sun [12] summarising the rainbow connection subject have also appeared recently.

Here, our aim is to extend the notion of rainbow connection to hypergraphs. Such an extension depends on the definition of a path in a hypergraph. To clarify this, we will actually consider two types of paths. For $\ell \geq 1$, a *Berge path*, or simply a *path*, is a hypergraph $\mathcal{P}$ consisting of a sequence $v_1, e_1, v_2, e_2, \dots, v_\ell, e_\ell, v_{\ell+1}$, where $v_1, \dots, v_{\ell+1}$ are distinct vertices, $e_1, \dots, e_\ell$ are distinct edges, and $v_i, v_{i+1} \in e_i$ for every $1 \leq i \leq \ell$. The *length* of a path is the number of its edges. If $\mathcal{H}$ is a connected hypergraph, then for $x, y \in V(\mathcal{H})$, an *x–y path* is a path with a sequence $v_1, e_1, \dots, v_\ell, e_\ell, v_{\ell+1}$, where $x = v_1$

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and $y = v_{\ell+1}$. The *distance* from x to y, denoted by $d(x, y)$, is the minimum possible length of an x–y path in $\mathcal{H}$. The *diameter* of $\mathcal{H}$ is $\mathrm{diam}(\mathcal{H}) = \max_{x,y\in V(\mathcal{H})} d(x, y)$.

For $\ell \geq 1$ and $1 \leq s < r$, an (r, s)-*path* is an r-uniform hypergraph $\mathcal{P}'$ with vertex set $V(\mathcal{P}') = \{v_1, \ldots, v_{(\ell-1)(r-s)+r}\}$ and edge set

$$E(\mathcal{P}') = \{v_1 \cdots v_r, v_{r-s+1} \cdots v_{r-s+r}, v_{2(r-s)+1} \cdots v_{2(r-s)+r}, \ldots, v_{(\ell-1)(r-s)+1} \cdots v_{(\ell-1)(r-s)+r}\}.$$

In other words, $\mathcal{P}'$ is an interval hypergraph where all the intervals have size r, and they can be linearly ordered so that every two consecutive intervals intersect in exactly s vertices. For a hypergraph $\mathcal{H}$ and $x, y \in V(\mathcal{H})$, an x–y (r, s)-*path* is an (r, s)-path as described above, with $x = v_1$ and $y = v_{(\ell-1)(r-s)+r}$, if such an (r, s)-path exists in $\mathcal{H}$. Let $\mathcal{F}_{r,s}$ be the family of the hypergraphs $\mathcal{H}$ such that, for every $x, y \in V(\mathcal{H})$, there exists an x–y (r, s)-path. Note that every member of $\mathcal{F}_{r,s}$ is connected. For $\mathcal{H} \in \mathcal{F}_{r,s}$ and $x, y \in V(\mathcal{H})$, the (r, s)-*distance* from x to y, denoted by $d_{r,s}(x, y)$, is the minimum possible length of an x–y (r, s)-path in $\mathcal{H}$. The (r, s)-*diameter* of $\mathcal{H}$ is $\mathrm{diam}_{r,s}(\mathcal{H}) = \max_{x,y\in V(\mathcal{H})} d_{r,s}(x, y)$. If an (r, s)-path has edges $e_1, \ldots, e_\ell$, then we will often write the (r, s)-path as $\{e_1, \ldots, e_\ell\}$.

The definition of Berge paths was introduced by Berge in the 1970's. The introduction of (r, s)-paths appeared more recently. Notably, in 1999, Katona and Kierstead [8] studied (r, s)-paths when they posed a problem concerning a generalisation of Dirac's theorem to hypergraphs, and since then, such paths have been well-studied.

An edge-coloured path or (r, s)-path (for $1 \leq s < r$) is *rainbow* if its edges have distinct colours. For a connected hypergraph $\mathcal{H}$, an edge-colouring of $\mathcal{H}$ is *rainbow connected* if for any two vertices $x, y \in V(\mathcal{H})$, there exists a rainbow x–y path. The *rainbow connection number* of $\mathcal{H}$, denoted by $rc(\mathcal{H})$, is the minimum integer t for which there exists a rainbow connected edge-colouring of $\mathcal{H}$ with t colours. Clearly, we have $rc(\mathcal{H}) \geq \mathrm{diam}(\mathcal{H})$. Similarly, for $\mathcal{H} \in \mathcal{F}_{r,s}$, an edge-colouring of $\mathcal{H}$ is (r, s)-*rainbow connected* if for any two vertices $x, y \in V(\mathcal{H})$, there exists a rainbow x–y (r, s)-path. The (r, s)-*rainbow connection number* of $\mathcal{H}$, denoted by $rc(\mathcal{H}, r, s)$, is the minimum integer t for which there exists an (r, s)-rainbow connected edge-colouring of $\mathcal{H}$ with t colours. Again, we have $rc(\mathcal{H}, r, s) \geq \mathrm{diam}_{r,s}(\mathcal{H})$. Also, note that for $n \geq r \geq 2$, we have $rc(\mathcal{K}_n^r) = rc(\mathcal{K}_n^r, r, s) = 1$, where $\mathcal{K}_n^r$ is the complete r-uniform hypergraph on n vertices.

Hence, we have two generalisations of the rainbow connection number from graphs to hypergraphs. There are good reasons to consider both generalisations. We consider the version with Berge paths because this covers the situation for a larger class of hypergraphs, namely, all connected hypergraphs, rather than just the class $\mathcal{F}_{r,s}$ for the (r, s)-paths version. On the other hand, for many hypergraphs, the version with the (r, s)-paths is more interesting than the one with the Berge paths, in the sense that $rc(\mathcal{H}, r, s)$ is much more difficult to determine than $rc(\mathcal{H})$.

This paper will be organised as follows. In Section 2, we shall give a characterisation of those hypergraphs $\mathcal{H}$ with $rc(\mathcal{H}) = e(\mathcal{H})$ and study $rc(\mathcal{H})$ and $rc(\mathcal{H}, r, s)$ for some specific hypergraphs, namely, cycles and complete multipartite hypergraphs. In Section 3, we will show that the functions $rc(\mathcal{H})$ and $rc(\mathcal{H}, r, s)$ (for $1 \leq s < r$ and $r \geq 3$) are separated in the following sense: there is an infinite family of hypergraphs $\mathcal{G} \subset \mathcal{F}_{r,s}$ such that, $rc(\mathcal{H})$ is bounded on $\mathcal{G}$ by an absolute constant—we will in fact show that $rc(\mathcal{H}) = 2$ on $\mathcal{G}$; and $rc(\mathcal{H}, r, s)$ is unbounded. Note that we have $rc(\mathcal{H}, r, s) \geq rc(\mathcal{H})$ for all $\mathcal{H} \in \mathcal{F}_{r,s}$. Similarly, we will show that the functions $rc(\mathcal{H}, r, s)$ and $rc(\mathcal{H}, r, s')$ (for $1 \leq s \neq s' < r$ and $r \geq 3$) are separated, by proving that $rc(\mathcal{H}, r, s) = 2$ and $rc(\mathcal{H}, r, s')$ is unbounded on an infinite family of hypergraphs $\mathcal{G} \subset \mathcal{F}_{r,s} \cap \mathcal{F}_{r,s'}$. Hence, a bound for one of $rc(\mathcal{H}, r, s)$ and $rc(\mathcal{H}, r, s')$ does not in general imply a bound for the other.

2. Rainbow connection of some hypergraphs

In [5, Proposition 1.1], Chartrand et al. proved that for a connected graph G, we have $rc(G) = e(G)$ if and only if G is a tree. We would like to say something similar for hypergraphs. That is, what is a necessary and sufficient condition for a hypergraph $\mathcal{H}$ to have $rc(\mathcal{H}) = e(\mathcal{H})$?

Recall that a hypergraph $\mathcal{T}$ is a *hypertree* if $\mathcal{T}$ is connected, and there exists a simple tree T with $V(T) = V(\mathcal{T})$, with the vertex set of every edge of $\mathcal{T}$ inducing a subtree of T. Unfortunately, in the hypergraphs setting, a necessary and sufficient condition on $\mathcal{H}$ for $rc(\mathcal{H}) = e(\mathcal{H})$ is *not* that $\mathcal{H}$ is a hypertree. There are infinitely many hypertrees $\mathcal{T}$ where $rc(\mathcal{T}) < e(\mathcal{T})$. For example, consider the hypertree $\mathcal{T}$ which is the (3, 2)-path of length ℓ, where $\ell \geq 3$. Let $e_1, \ldots, e_\ell$ be the consecutive edges of $\mathcal{T}$. By assigning distinct colours to the edges

$$\begin{cases} e_1, e_3, e_5, \ldots, e_\ell & \text{if } \ell \text{ is odd,} \\ e_1, e_3, e_5, \ldots, e_{\ell-1}, e_\ell & \text{if } \ell \text{ is even,} \end{cases}$$

and then arbitrary (used) colours to the remaining edges, we have a rainbow connected edge-colouring for $\mathcal{T}$, and $rc(\mathcal{T}) \leq \lfloor \frac{\ell}{2} \rfloor + 1$. In fact, we have $rc(\mathcal{T}) = \lfloor \frac{\ell}{2} \rfloor + 1$, since $rc(\mathcal{T}) \geq \mathrm{diam}(\mathcal{T}) = \lfloor \frac{\ell}{2} \rfloor + 1$. Hence, we have $rc(\mathcal{T}) = \lfloor \frac{\ell}{2} \rfloor + 1 < \ell = e(\mathcal{T})$.

Nevertheless, we can still find such a necessary and sufficient condition, which will be a connectivity property. Recall that a graph G with $e(G) \geq 1$ is *minimally connected* if G is connected, and for every $e \in E(G)$ the graph $(V(G), E(G) \setminus \{e\})$ is disconnected. It is well-known that if $e(G) \geq 1$, then G is minimally connected if and only if G is a tree. Hence, Chartrand et al.'s result can be restated as follows: "For a connected graph G with $e(G) \geq 1$, we have $rc(G) = e(G)$ if and only if G is minimally connected". In this direction, we *do* have the analogous situation for hypergraphs.

We say that a hypergraph $\mathcal{H}$ with $e(\mathcal{H}) \geq 1$ is *minimally connected* if $\mathcal{H}$ is connected, and for every $e \in E(\mathcal{H})$, the hypergraph $(V(\mathcal{H}), E(\mathcal{H}) \setminus \{e\})$ is disconnected. Note that, unlike in the graphs setting, hypertrees and minimally