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Graphs whose complement and square are isomorphic

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ABSTRACT

We study square-complementary graphs, that is, graphs whose complement and square are isomorphic. We prove several necessary conditions for a graph to be square-complementary, describe ways of building new square-complementary graphs from existing ones, construct infinite families of square-complementary graphs, and characterize square-complementary graphs within various graph classes. The bipartite case turns out to be of particular interest. We also exhibit a square-complementary graph with a triangle, thus answering in the affirmative a question asked independently by Baltić et al. and by Capobianco and Kim.

1. Introduction

A graph is self-complementary (s.c.) if it is isomorphic to its complement. The study of self-complementary graphs was initiated by Sachs [37] and later, but independently, by Ringel [36]. Self-complementary graphs have been studied extensively in the literature. There exist small self-complementary graphs, for example the one-vertex graph K_1, the 4-vertex path P_4, and the 5-vertex cycle C_5. Moreover, it is known that there exists a self-complementary graph on n vertices if and only if $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

Given two graphs G and H, we say that G is the *square* of H (and denote this by $G = H^2$) if their vertex sets coincide and two distinct vertices x, y are adjacent in G if and only if x, y are at distance at most two in H. A number of results about squares of graphs and their properties are available in the literature (see, e.g., Section 10.6 in the monograph [9]).

The aim of this paper is to study the following notion, similar to that of self-complementary graphs.

Definition 1.1. A graph G is said to be *square-complementary* (*squco* for short) if its square is isomorphic to its complement. That is, $G^2 \cong \overline{G}$, or, equivalently, $G \cong \overline{G^2}$.

The question of characterizing squco graphs was posed by Seymour Schuster at a conference in 1980 [38]. Since then, squco graphs were studied in the context of graph equations in terms of operators such as the line graph and complement

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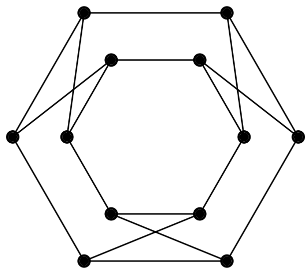

Fig. 1. The Franklin graph.

(see [2,5,12,11,13,34]). The entire set of solutions of some of these equations was found (see for example [2] and references quoted therein). However, the set of solutions of the equation $G^2 \cong \overline{G}$ remains unknown despite several attempts to describe it (see for example [5,12]). The problem of determining all squco graphs was also posed as Open Problem No. 36 in Prisner's book [34].

Examples of squco graphs are K_1, C_7, and more generally, the graph obtained from the 7-cycle by replacing one of its vertices, say v, by arbitrarily many pairwise non-adjacent vertices adjacent precisely to the two neighbors of v. This construction [2] (see also [5]) implies the following.

Proposition 1.1. *For every integer $n \geq 7$, there exists a squco graph with n vertices.*

Capobianco and Kim conjectured in [12] that every squco graph of diameter 3 contains C_7 as an induced subgraph. This conjecture is false, as shown by a cubic vertex–transitive bipartite squco graph on 12 vertices, known as the Franklin graph. A drawing of it is depicted in Fig. 1.

All the examples above together with the circulant graph $C_{17}(\{1, 4\})$ (see Section 2 for definition) were mentioned already by Akiyama et al. [2]. The Franklin graph also appeared in the context of self-complementary bipartite graphs in [1,22].

Baltić et al. [5] proved that every nontrivial squco graph is connected, has no cut vertices, and every vertex is of eccentricity either 3 or 4 (which was already observed by Capobianco and Kim [12]). Baltić et al. also showed that every nontrivial squco graph has radius 3, and regular squco graphs have diameter 3. In general, squco graphs can have diameter either 3 or 4 (although no squco graph with diameter 4 is known). The fact that no nontrivial squco graph has a cut vertex implies that the only squco tree is K_1. In addition, Baltić et al. provided a way of constructing arbitrarily large squco graphs from smaller ones (see Section 2). Capobianco, Losi and Riley [13] defined $D(i, j)$-balanced graphs and proved that every squco graph is $D(1, 3)$-balanced. Capobianco and Kim further strengthened this result in terms of a condition based on degree sequences [12]. Capobianco and Kim also gave upper and lower bounds on the number of edges in diameter 3 squco graphs, and gave upper bounds on the clique number in terms of the number of vertices.

In Section 2, we revise a previously known construction of new squco graphs from existing ones, and examine several of its consequences. We also give a sufficient condition for constructing a squco graph by adding a vertex to another squco graph. We provide some further examples of squco graphs, that, in particular, answer in the affirmative a question asked independently by Baltić et al. and by Capobianco and Kim whether there exists a squco graph with a triangle. In Section 3 we prove some necessary conditions for graphs to be squco. Section 4 is devoted to characterizations of squco graphs in particular graph classes. We consider several subclasses of perfect graphs, graphs of independence number at most three, graphs of maximum degree at most three, and graphs on at most 11 vertices. In Section 5, we give a characterization of bipartite squco graphs, and exemplify it with a construction (based on bipartite bi-Cayley graphs over cyclic groups) of an infinite family of bipartite vertex–transitive squco graphs generalizing the Franklin graph. We conclude the paper with a discussion on possible generalizations and open problems in Section 6.

We briefly recall some useful definitions. We denote by P_n, C_n and K_n the path, the cycle, and the complete graph on n vertices, respectively, and $2K_2$ denotes the disjoint union of two K_2's. The *closed neighborhood* of a vertex v in a graph G is the set $N_G[v]$ consisting of the neighborhood $N_G(v)$ of v together with v. For a positive integer i, we denote by $N_i(v, G)$ the set of all vertices u in G such that $d_G(u, v) = i$, and by $N_{\geq i}(v, G)$ the set of all vertices u in G such that $d_G(u, v) \geq i$. By $B_i(v, G)$ (or just $B_i(v)$ if the graph is clear from the context), we denote the ball of radius i centered at v, that is, the set of all vertices u in G such that $d_G(u, v) \leq i$. The *eccentricity* $\mathrm{ecc}_G(u)$ of a vertex u in a graph G is the maximum of the numbers $d_G(u, v)$ where $v \in V(G)$. Unless stated otherwise we use standard graph terminology, referring the reader to [16] (see also [28]).

2. Examples and constructions

In this section, we provide some further examples of squco graphs, and discuss ways of building new squco graphs from known ones.

As noted already in the introduction, the 7-cycle is a squco graph. It is also a *circulant*, that is, a Cayley graph over a cyclic group. Further examples of squco graphs can be found among circulants. For an integer $n \geq 3$ and $D \subseteq [\lfloor \frac{n}{2} \rfloor] := \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$, the *circulant graph* $C_n(D)$ is the graph with vertex set $[0, n-1] := \{0, 1, \dots, n-1\}$ in which two vertices