Discrete Mathematics 327 (2014) 103–117

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Extremal statistics on non-crossing configurations[☆]

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ARTICLE INFO

Article history:
Received 29 March 2012
Received in revised form 12 March 2014
Accepted 18 March 2014
Available online 3 April 2014

Keywords:
Noncrossing configuration
Extremal parameter
Analytic combinatorics

ABSTRACT

We analyze extremal statistics in non-crossing configurations on the n vertices of a convex polygon. We prove that the maximum degree and the largest component are of logarithmic order, and that, suitably scaled, they converge to a well-defined constant. We also prove that the diameter is of order $\sqrt{n}$. The proofs are based on singularity analysis, an application of the first and second moment method, and on the analysis of iterated functions.

1. Introduction and preliminaries

Let $p_1, \ldots, p_n$ be the vertices of a convex polygon in the plane, labeled counterclockwise. A *non-crossing graph* (or configuration) is a graph on these vertices such that when the edges are drawn as straight lines the only intersections occur at vertices. The *root* of a graph is vertex p_1. We call the edge p_1p_n (if present) the root edge.

From now on, all graphs are assumed to be non-crossing graphs. A triangulation is a graph with the maximum number of edges and it is characterized by the fact that all internal faces are triangles. A dissection is a graph containing all the boundary edges $p_1p_2, p_2p_3, \ldots, p_np_1$; a single edge p_1p_2 is also considered a dissection (see Fig. 1). From a graph theoretic point of view, dissections are the same as 2-connected graphs. The root region of a dissection is the internal region adjacent to the root edge.

The enumerative theory of non-crossing configurations is an old subject, going back to Euler; see, for instance, Comtet's book [4] for an account of classical results. Flajolet and Noy [11] reexamined these problems using the tools from analytic combinatorics [13] in a unified way. They showed that for all natural classes under consideration the number of non-crossing graphs with n vertices is asymptotically of the form

$$cn^{-3/2}\gamma^n,$$

for some positive constants c and γ. In addition, many basic parameters obey a Gaussian limit law with linear expectation and variance. These include: number of edges, number of components, number of leaves in trees and number of blocks in partitions. The proofs in [11] are based on perturbation of singularities and extensions of the Central Limit Theorem.

In this paper we take a step further and analyze more complex parameters, specially *extremal* parameters. Some results have been obtained previously for triangulations [6,14] and trees [5,16], but here we aim at a systematic treatment of the subject, covering the most important extremal parameters and proving limit laws whenever possible. Our main results are summarized as follows.

☆ Research partially supported by Projects MTM2011-24097 and DGR2009-SGR1040.

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http://dx.doi.org/10.1016/j.disc.2014.03.016

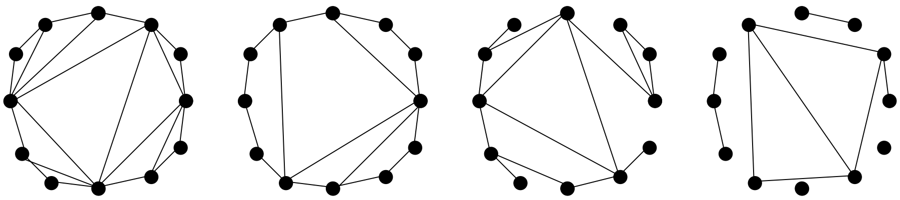

Fig. 1. From left to right: a triangulation, a 2-connected graph (dissection), a connected graph, and an arbitrary graph.

- For graphs, connected graphs and 2-connected graphs, the degree of the root vertex converges to a discrete law. More precisely, if p_k is the probability that the root has degree k, then
$$\sum p_k w^k = \frac{A(w)}{(1-qw)^2},$$
where $A(w)$ is a polynomial of degree two and q is a quadratic irrational with $0 < q < 1$. It follows that the tail of the distribution satisfies, for a suitable constant $c > 0$,
$$p_k \sim ckq^k, \quad \text{as } k \to \infty.$$
- For graphs, connected graphs and 2-connected graphs, the maximum degree Δ_n is of logarithmic order. More precisely, for each class under consideration there exists a quadratic irrational $q > 0$ such that
$$\frac{\Delta_n}{\log n} \to \frac{1}{\log(q^{-1})} \quad \text{in probability.}$$
- The largest connected component M_n in graphs is of logarithmic order: there exists a quadratic irrational $q > 0$ such that
$$\frac{M_n}{\log n} \to \frac{1}{\log(q^{-1})} \quad \text{in probability.}$$
- For triangulations, connected graphs and 2-connected graphs, the diameter D_n is of order $\sqrt{n}$. For each class under consideration, there exist constants $0 < c_1 < c_2$ such that
$$c_1\sqrt{n} < \mathbb{E}D_n < c_2\sqrt{n}.$$

These results reflect the tree-like nature of non-crossing configurations. In particular, the diameter is of order $\sqrt{n}$, like the height of plane trees. The expected maximum degree in triangulations was shown to be asymptotically $\log n/\log 2$ in [6] (much more precise results were obtained in [14]). To our knowledge, the diameter of random configurations has not been studied before, even in the basic case of triangulations.

In the rest of this section we collect several technical preliminaries needed in the paper. In Section 2 we analyze the degree of the root vertex. Sections 3 and 4 are devoted to the maximum degree and the size of the largest component, respectively, and are based on the first and second moment method. Finally, in Section 5 we analyze the diameter, making use of iterated functions.

1.1. Generating functions

We denote by $G(z)$ and $C(z)$ the generating functions for arbitrary and connected graphs, respectively, counted by the number of vertices. Furthermore, let $B(z)$ be the generating function for 2-connected graphs, where z marks the number of vertices minus one. We have the following relations for the generating functions. The first one reflects the decomposition of a dissection as a sequence of dissections attached to the root region, as in [11]. See Fig. 2 (left) for an illustration.

$$B(z) = z + \frac{B(z)^2}{1-B(z)}. \tag{1}$$

Of the two solutions of this equation, only one is a power series, hence

$$B(z) = \frac{1+z-\sqrt{1-6z+z^2}}{4} \tag{2}$$

which has a square-root singularity at $z = 3-2\sqrt{2}$. The next equation encodes the decomposition of a connected graph into 2-connected components, also called blocks; see Fig. 2 (center).

$$C(z) = \frac{z}{1-B(C(z)^2/z)}. \tag{3}$$

Indeed, a connected graph consists of a root and a sequence of blocks containing the root, in which each vertex is substituted by a pair of connected graphs (to the left and to the right) with one vertex identified. Eliminating B we obtain

$$C(z)^3 + C(z)^2 - 3zC(z) + 2z^2 = 0, \tag{4}$$