Discrete Mathematics 327 (2014) 118–129

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Counting proper mergings of chains and antichains

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ARTICLE INFO

Article history:
Received 3 August 2012
Received in revised form 18 March 2014
Accepted 21 March 2014
Available online 3 April 2014

Keywords:
Poset counting
Distributive lattices
Plane partitions
Monotone coloring
Narayana numbers

ABSTRACT

A proper merging of two disjoint quasi-ordered sets $(P, \leftarrow_{\mathcal{P}})$ and $(Q, \leftarrow_{\mathcal{Q}})$, denoted by $\mathcal{P}$ and $\mathcal{Q}$, respectively, is a quasi-order on the union of P and Q such that the restriction to P or Q yields the original quasi-orders on those sets and such that no elements of P and Q are identified. In this article, we consider the cases where $\mathcal{P}$ and $\mathcal{Q}$ are chains, where $\mathcal{P}$ and $\mathcal{Q}$ are antichains, and where $\mathcal{P}$ is an antichain and $\mathcal{Q}$ is a chain. We give formulas that determine the number of proper mergings in all three cases. We also introduce two new bijections from proper mergings of two chains to plane partitions and from proper mergings of an antichain and a chain to monotone colorings of complete bipartite digraphs.

1. Introduction

Let $(P, \leftarrow_{\mathcal{P}})$ and $(Q, \leftarrow_{\mathcal{Q}})$ be two disjoint quasi-ordered sets, denoted by $\mathcal{P}$ and $\mathcal{Q}$, respectively. A *merging* of $\mathcal{P}$ and $\mathcal{Q}$ is a quasi-ordered set $(P \cup Q, \leftarrow)$ such that the restriction of $\leftarrow$ to P or Q yields $\leftarrow_{\mathcal{P}}$ or $\leftarrow_{\mathcal{Q}}$, respectively. In other words, a merging of $\mathcal{P}$ and $\mathcal{Q}$ is determined by a quasi-order on the union of P and Q that does not change the quasi-orders on P and Q.

In [4], a characterization of the set of mergings of two arbitrary quasi-ordered sets $\mathcal{P}$ and $\mathcal{Q}$ is given. In particular, it turns out that every merging of $\mathcal{P}$ and $\mathcal{Q}$ can be uniquely described by two binary relations $R \subseteq P \times Q$ and $S \subseteq Q \times P$. The relation R can be interpreted as a description of which part of $\mathcal{P}$ is weakly below $\mathcal{Q}$, and analogously the relation S can be interpreted as a description of which part of $\mathcal{Q}$ is weakly below $\mathcal{P}$. A merging is *proper* if $R \cap S^{-1} = \emptyset$, and hence if no element of P is identified with an element of Q.

The characterization in [4] uses techniques of Formal Concept Analysis (FCA, see [5] for an introduction), a branch of mathematics that investigates binary relations, the so-called "formal contexts", between two sets. The starting point of FCA is the construction of a closure system from such a formal context. Then, this closure system induces a complete lattice when ordering the closed sets by inclusion. The basic theorem of FCA states that every complete lattice can be derived from a formal context. Theorem 1 in [4] states that the set of mergings of two quasi-ordered sets $\mathcal{P}$ and $\mathcal{Q}$ forms a complete distributive lattice, and can thus be described by a formal context. Notably, this formal context can be constructed easily from the quasi-orders $\leftarrow_{\mathcal{P}}$ and $\leftarrow_{\mathcal{Q}}$. The set of proper mergings of $\mathcal{P}$ and $\mathcal{Q}$ constitutes a complete distributive sublattice of the previous lattice.

Unfortunately, the formal context provides very little information about the size of its associated lattice. Hence, although the set of mergings of two quasi-ordered sets $\mathcal{P}$ and $\mathcal{Q}$ can be described completely, not much is known about its size. This article provides a first enumerative analysis of the set of proper mergings of two special classes of quasi-ordered sets, namely chains and antichains. The actual genesis of this article was the observation that the number of proper mergings of two n-chains is given by

$$F_c(n) = \frac{(2n)!(2n+1)!}{(n!(n+1)!)^2}.$$

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http://dx.doi.org/10.1016/j.disc.2014.03.020

It is stated in [2] that $F_c(n)$ also determines the number of plane partitions with n rows, n columns and largest part at most 2. See [11, Sequence A000891] for some other objects counted by this number. It is not hard to define a bijection from the set of these plane partitions to the set of proper mergings of two n-chains, as will be described in Section 3. It is then straightforward to extend this bijection to map from the set of plane partitions with m rows, n columns and largest part at most 2 to the set of proper mergings of an m-chain and an n-chain. Since the number of such plane partitions can be derived from MacMahon's formula, see (6), this bijection easily allows for counting the proper mergings of two chains. As a corollary, we obtain that the number of proper mergings of an m-chain and an n-chain is the Narayana number $\mathrm{Nar}(m+n+1, m+1)$, see [10]. Thus our bijection relates the proper mergings of two chains to a wealth of well-known mathematical objects such as trees with a certain number of leaves, or Dyck paths with a certain number of peaks. See Remark 3.6 for further details.

After succeeding in enumerating proper mergings of chains, we became curious whether we can count proper mergings of two antichains in a similar way. Unfortunately, we cannot give a bijection from the set of proper mergings of two antichains to any other known mathematical set. However, we are able to enumerate the proper mergings of two antichains with the help of a generating function found by Christian Krattenthaler. See Section 4 for the details.

The third part of this article is devoted to the enumeration of proper mergings of an m-antichain and an n-chain. When computing the number of these proper mergings with the help of Daniel Borchmann's FCA-tool CONEXP-CLJ [1], we recovered [11, Sequence A085465]. The formula generating this sequence is a special case of the following formula:

$$F_{ac}(m,n) = \sum_{i=1}^{n+1}\Big((n+2-i)^m - (n+1-i)^m\Big)\cdot i^m.$$

It is stated in [6, Proposition 4.5] that $F_{ac}(m,n)$ also equals the number of monotone $(n+1)$-colorings of the complete bipartite digraph $\vec{K}_{m,m}$. In Section 5, we construct a bijection from the set of proper mergings of an m-antichain and an n-chain to the set of monotone $(n+1)$-colorings of $\vec{K}_{m,m}$.

The precise statements of the results described in the previous paragraphs are collected in the following theorem.

Theorem 1.1. *Let $\mathfrak{M}^{\bullet}_{\mathcal{P},\mathcal{Q}}$ denote the set of proper mergings of two quasi-ordered sets $(P, \leftarrow_{\mathcal{P}})$ and $(Q, \leftarrow_{\mathcal{Q}})$, denoted by $\mathcal{P}$ and $\mathcal{Q}$, respectively.*

(i) *Let $\mathcal{P}$ and $\mathcal{Q}$ be chains. If $|P| = m$ and $|Q| = n$, then*

$$\left|\mathfrak{M}^{\bullet}_{\mathcal{P},\mathcal{Q}}\right| = \frac{1}{n+m+1}\binom{n+m+1}{m+1}\binom{n+m+1}{m}.$$

(ii) *Let $\mathcal{P}$ and $\mathcal{Q}$ be antichains. If $|P| = m$ and $|Q| = n$, then*

$$\left|\mathfrak{M}^{\bullet}_{\mathcal{P},\mathcal{Q}}\right| = \sum_{n_1+m_1+k_1=m}\binom{m}{n_1, m_1, k_1}(-1)^{k_1}\Big(2^{n_1}+2^{m_1}-1\Big)^n.$$

(iii) *Let $\mathcal{P}$ be an antichain, and let $\mathcal{Q}$ be a chain. If $|P| = m$ and $|Q| = n$, then*

$$\left|\mathfrak{M}^{\bullet}_{\mathcal{P},\mathcal{Q}}\right| = \sum_{i=1}^{n+1}\Big((n+2-i)^m - (n+1-i)^m\Big)\cdot i^m.$$

In Theorem 1.1(iii), we need to be careful with the case $m = 0$, since a term of the form "0^0" appears in the sum. Since there is exactly one proper merging of an empty antichain and some chain, we need to interpret this term as being equal to zero.

This article is organized as follows: in Section 2, we introduce the necessary order-theoretic notions and formally define mergings of two quasi-ordered sets. In Section 3, we define a bijection from proper mergings of two chains to plane partitions with largest part 2, and we obtain Theorem 1.1(i) as a consequence of this bijection. In Section 4, we compute the generating function for the proper mergings of two antichains and obtain Theorem 1.1(ii). Finally, in Section 5, we construct a bijection from proper mergings of an antichain and a chain to monotone colorings of a complete bipartite digraph, and we obtain Theorem 1.1(iii) as a consequence of this bijection.

2. Preliminaries

In this section, we introduce the necessary order-theoretic notions and concepts. For a more detailed introduction, we refer for instance to [3].

Let P be a set. A *quasi-order on* P is a binary relation on P that is transitive and reflexive; it will usually be denoted by $\leftarrow$. A quasi-order on P that in addition is antisymmetric is a *partial order on* P; it will usually be denoted by $\leq$. Now, let $(P, \leftarrow)$ be a quasi-ordered set, denoted by $\mathcal{P}$. We call a set $X \subseteq P$ a *down-set of* $\mathcal{P}$ if for every $x \in X$ and every $p \in P$ with $p \leftarrow x$ it follows that $p \in X$. Dually, a set $X \subseteq P$ is an *up-set of* $\mathcal{P}$ if for every $x \in X$ and every $p \in P$ with $x \leftarrow p$ it follows that $p \in X$. For any binary relation $R \subseteq P \times P$, we can define the *inverse relation* by $R^{-1} = \{(p', p) \mid (p, p') \in R\}$. The pair (P, R^{-1}) is the *dual of* (P, R), and will be denoted by $(P, R)^d$.