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New large sets of resolvable Mendelsohn triple systems[☆]

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ABSTRACT

An LRMTS(v) is a large set consisting of $v-2$ pairwise disjoint resolvable Mendelsohn triple systems defined over the same v-element set. In this paper a new product construction for LRMTS is displayed by using generalized LR-designs. As well, several new existence families of LRMTS are constructed via 3-wise balanced designs. Finally the existence result of LRMTS is expanded by combining all the known constructions.

1. Introduction

The study of large sets constitutes an important part of combinatorial design theory. The remarkable work by Lu [19,20] and Teirlinck [22] on the existence problem of large sets of Steiner triple systems inspired many researchers to investigate large sets of varied forms. Many researchers devoted to the study of large sets of t-designs where $t \geq 3$, see [17] for a survey. Meanwhile, some researchers put efforts into large sets of oriented triple systems, maybe further imposing certain restricted property, such as resolvable restriction, with which we are concerned in this paper. Although the most classical existence problem of large sets of Kirkman triple systems (LKTS) remains still very much open, a number of new constructions for LKTS were developed, especially in the last two decades, see [3] for a recent survey. Enlightened by these approaches, we investigate the problem of large sets of resolvable Mendelsohn triple systems (LRMTS) and provide some new infinite existence families. We now give the definition of LRMTS and include some known results.

Let X be a finite set of v elements. An *ordered pair* of X is always a pair (x, y) with $x, y \in X$ and $x \neq y$. A *cyclic triple* on X is a set of three ordered pairs (x, y), (y, z) and (z, x) of X, which is denoted by $\langle x, y, z\rangle$ (or $\langle y, z, x\rangle$, or $\langle z, x, y\rangle$). A *Mendelsohn triple system* of order v, briefly by MTS(v), is a pair $(X, \mathcal{B})$ where $\mathcal{B}$ is a collection of cyclic triples on X, called *blocks*, such that every ordered pair of X belongs to exactly one block of $\mathcal{B}$.

An MTS(v)$(X, \mathcal{B})$ is called *resolvable* if its block set $\mathcal{B}$ can be partitioned into subsets (called *parallel classes*), each containing every element of X exactly once. A resolvable MTS(v) is denoted by RMTS(v). An RMTS(v) exists if and only if $v \equiv 0 \pmod 3$ and $v \neq 6$, see [1].

A *large set* of MTS(v)s, denoted by LMTS(v), is a collection $\{(X, \mathcal{B}_i)\}$, where every $(X, \mathcal{B}_i)$ is an MTS(v) and all $\mathcal{B}_i$'s form a partition of all cyclic triples on X. It is easy to see that an LMTS(v) consists of $v-2$ pairwise disjoint MTS(v)s. An LMTS(v) exists if and only if $v \equiv 0, 1 \pmod 3$ and $v \geq 3$, $v \neq 6$ [11,12,18]. An LRMTS(v) denotes an LMTS(v) in which each member MTS(v) is resolvable.

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We now illustrate a direct construction for the smallest possible even order of LRMTS, that is, an LRMTS(12), which also implies another design we will use in one of the constructions in this paper.

Example 1.1 (*[15, Lemma 2.3]*)**.** There exists an LRMTS(12).

Construction. Let $X = (\mathbb{Z}_5 \cup \{\infty\}) \times I_2$ be the point set. An LRMTS(12) consists of ten pairwise disjoint RMTS(12)s, which are developed from two base RMTS(12)s under the action of $\mathbb{Z}_5$. Firstly we construct twelve parallel classes of X and divide them into two parts.

$$
\begin{array}{lllll}
P_1^0: & \langle(3,0),(4,0),(\infty,0)\rangle & \langle(1,0),(1,1),(4,1)\rangle & \langle(2,0),(2,1),(3,1)\rangle & \langle(0,0),(0,1),(\infty,1)\rangle \\
P_2^0: & \langle(3,1),(0,1),(4,1)\rangle & \langle(1,1),(1,0),(4,0)\rangle & \langle(2,1),(2,0),(3,0)\rangle & \langle(\infty,1),(\infty,0),(0,0)\rangle \\
P_3^0: & \langle(2,0),(1,1),(\infty,1)\rangle & \langle(4,1),(4,0),(1,0)\rangle & \langle(3,0),(3,1),(2,1)\rangle & \langle(0,1),(0,0),(\infty,0)\rangle \\
P_4^0: & \langle(2,1),(0,0),(1,0)\rangle & \langle(4,0),(4,1),(1,1)\rangle & \langle(3,1),(3,0),(2,0)\rangle & \langle(\infty,0),(\infty,1),(0,1)\rangle \\
P_5^0: & \langle(1,0),(3,0),(\infty,0)\rangle & \langle(1,1),(0,1),(3,1)\rangle & \langle(4,0),(2,1),(\infty,1)\rangle & \langle(4,1),(0,0),(2,0)\rangle \\
P_8^0: & \langle(3,1),(4,0),(\infty,1)\rangle & \langle(2,0),(0,1),(1,0)\rangle & \langle(2,1),(1,1),(\infty,0)\rangle & \langle(3,0),(0,0),(4,1)\rangle \\
P_1^1: & \langle(3,1),(4,1),(\infty,1)\rangle & \langle(1,0),(1,1),(4,0)\rangle & \langle(2,0),(2,1),(3,0)\rangle & \langle(0,0),(0,1),(\infty,0)\rangle \\
P_2^1: & \langle(3,0),(0,0),(4,0)\rangle & \langle(1,1),(1,0),(4,1)\rangle & \langle(2,1),(2,0),(3,1)\rangle & \langle(\infty,1),(\infty,0),(0,1)\rangle \\
P_3^1: & \langle(2,1),(1,0),(\infty,0)\rangle & \langle(4,1),(4,0),(1,1)\rangle & \langle(3,0),(3,1),(2,0)\rangle & \langle(0,1),(0,0),(\infty,1)\rangle \\
P_4^1: & \langle(2,0),(0,1),(1,1)\rangle & \langle(4,0),(4,1),(1,0)\rangle & \langle(3,1),(3,0),(2,1)\rangle & \langle(\infty,0),(\infty,1),(0,0)\rangle \\
P_5^1: & \langle(1,1),(3,1),(\infty,1)\rangle & \langle(1,0),(0,0),(3,0)\rangle & \langle(4,1),(2,0),(\infty,0)\rangle & \langle(4,0),(0,1),(2,1)\rangle \\
P_8^1: & \langle(3,0),(4,1),(\infty,0)\rangle & \langle(2,1),(0,0),(1,1)\rangle & \langle(2,0),(1,0),(\infty,1)\rangle & \langle(3,1),(0,1),(4,0)\rangle.
\end{array}
$$

For $i = 1, 2$ and $j = 0, 1$, let

$$P_{5+i}^j = \{\langle(2^i \cdot x, u), (2^i \cdot y, v), (2^i \cdot z, w)\rangle : \langle(x, u), (y, v), (z, w)\rangle \in P_5^j\},$$

where the calculation is reduced in $\mathbb{Z}_5$ and $x \cdot \infty = \infty$ for any $x \in \mathbb{Z}_5$.

Similarly, for $i = 1, 2, 3$ and $j = 0, 1$, let

$$P_{8+i}^j = \{\langle(2^i \cdot x, u), (2^i \cdot y, v), (2^i \cdot z, w)\rangle : \langle(x, u), (y, v), (z, w)\rangle \in P_8^j\}.$$

Then define $\mathcal{B}_j = \bigcup_{i=1}^{11} P_i^j$ for $j = 0, 1$. It is not difficult to check that each of $(X, \mathcal{B}_j)$ $(j = 0, 1)$ forms an RMTS(12). Finally developing $\mathcal{B}_0$ and $\mathcal{B}_1$ under $(\mathbb{Z}_5, +)$ yields an LRMTS(12). □

Up to now, the known odd orders of LRMTS are mostly derived from LKTS and some direct constructions are given in [10,16]. For even orders, we refer to [15] for a few direct constructions using finite fields. The presently known recursive constructions are very limited, including a tripling construction and a product construction, see [2,23]. In the following lemma, we include a few known results of LRMTS.

Lemma 1.2. *There exists an* LRMTS$(3v)$, *where*

(1) [3,10,15,16] $v \in \{1, 4, 7, 8, 11, 13, 16, 20, 21, 23, 25, 28, 32, 35, 37, 40, 41, 43, 47, 53, 55, 61, 65, 67, 68, 71, 76, 77, 91, 92, 93, 95, 97, 100, 103, 113, 121\} \bigcup \{2^{2g+1}25^h + 1 : g + h \geq 1, g, h \geq 0\}$,
(2) [13,14] $3v = 7^n + 2,\ 13^n + 2,\ 25^n + 2,\ 2^{4n} + 2$, *or* $2^{6n} + 2$ *for* $n \geq 1$, *or*
(3) [3,25] v *is the product of some elements in* $L = \{4^m 25^n - 1 : m \geq 1, n \geq 0\} \bigcup \{4 \times 7^s - 1 : s \geq 0\} \bigcup \{2q^t + 1 : t \geq 0, q \equiv 7 \pmod{12}$ *and q is a prime power*$\}$.

In this paper we utilize partitionable Mendelsohn candelabra systems with certain resolvable property to construct LRMTS, which are introduced as preliminaries in Section 2. A new product construction is displayed in Section 3 to make it possible to extend the known designs to produce new ones. In Section 4 several new existence families of LRMTS are constructed via 3-wise balanced designs. Finally in Section 5 we combine the recursive constructions with the known LRMTS to reach a conclusion.

2. Partitionable Mendelsohn candelabra systems with resolvable properties

The partitionable candelabra system is an important design to produce large sets of Steiner triple systems, see [7]. As well, partitionable candelabra systems with certain specified resolvable property are employed extensively to construct large sets of Kirkman triple systems, see for instance [3]. We in this paper impose necessary resolvable restriction to partitionable Mendelsohn candelabra systems to form large sets of resolvable Mendelsohn triple systems.

Let X be a v-element set and K be a set of positive integers. A t-*wise balanced design* (t-BD) of order v is a pair $(X, \mathcal{A})$ where $\mathcal{A}$ is a family of subsets of X (called blocks) such that each t-element subset of X is contained in exactly one block of $\mathcal{A}$. An $S(t, K, v)$ denotes a t-BD of order v with block sizes from the set K. The notation $S(t, k, v)$ is often used for $K = \{k\}$. An $S(2, 3, v)$ is a *Steiner triple system of order* v, or STS(v).