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# More on the Terwilliger algebra of Johnson schemes

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ABSTRACT

In Levstein and Maldonado (2007), the Terwilliger algebra of the Johnson scheme $J(n, d)$ was determined when $n \geq 3d$. In this paper, we determine the Terwilliger algebra $\mathcal{T}$ for the remaining case $2d \leq n < 3d$.

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## 1. Introduction

Let $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ denote a commutative association scheme, where $X$ is a finite set. Suppose $\mathrm{Mat}_X(\mathbb{C})$ denotes the algebra over $\mathbb{C}$ consisting of all matrices whose rows and columns are indexed by $X$. For each $i$, let $A_i$ denote the binary matrix in $\mathrm{Mat}_X(\mathbb{C})$ whose $(x, y)$-entry is 1 if and only if $(x, y) \in R_i$. We call $A_i$ the $i$th *adjacency matrix* of $\mathcal{X}$. We abbreviate $A = A_1$, and call it the *adjacency matrix* of $\mathcal{X}$. The subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ spanned by $A_0, A_1, \ldots, A_d$ is called the *Bose–Mesner algebra* of $\mathcal{X}$, denoted by $\mathcal{B}$. Since $\mathcal{B}$ is commutative and generated by real symmetric matrices, it has a basis consisting of primitive idempotents, denoted by $E_0 = \frac{1}{|X|}J, E_1, E_2, \ldots, E_d$. For each $i \in \{0, 1, \ldots, d\}$, write

$$A_i = \sum_{j=0}^{d} p_i(j)E_j, \quad E_i = \frac{1}{|X|}\sum_{j=0}^{d} q_i(j)A_j.$$

The scalars $p_i(j)$ and $q_i(j)$ are called the *eigenvalues* and the *dual eigenvalues* of $\mathcal{X}$, respectively.

Fix $x \in X$. For $0 \leq i \leq d$, let $E_i^*$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ whose $(y, y)$-entry is defined by

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } (x, y) \in R_i, \\ 0, & \text{otherwise.} \end{cases}$$

The subalgebra $\mathcal{T}(x)$ of $\mathrm{Mat}_X(\mathbb{C})$ generated by $A_0, A_1, \ldots, A_d; E_0^*, E_1^*, \ldots, E_d^*$ is called the *Terwilliger algebra* of $\mathcal{X}$ with respect to $x$.

Terwilliger [12] first introduced the Terwilliger algebra of association schemes, which is an important tool in considering the structure of an association scheme. For more information, see [4,5,13,14]. The Terwilliger algebra is a finite-dimensional semisimple $\mathbb{C}$-algebra; it is difficult to determine its structure in general. The structures of the Terwilliger algebras of some

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association schemes have been determined; see [1,3] for group schemes, [15] for strongly regular graphs, [7,11] for Hamming schemes, [10] for Johnson schemes, [8] for odd graphs, and [9] for incidence graphs of Johnson geometry.

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and $\binom{[n]}{d}$ denote the collection of all $d$-element subsets of $[n]$. For $0 \le i \le d$, define $R_i = \{(x, y) \in \binom{[n]}{d} \times \binom{[n]}{d} \mid |x \cap y| = d - i\}$. Then $(\binom{[n]}{d}, \{R_i\}_{i=0}^{d})$ is a symmetric association scheme of class $d$, which is called the *Johnson scheme*, denoted by $J(n, d)$. Note that $J(n, d)$ is isomorphic to $J(n, n - d)$. So we always assume that $n \ge 2d$.

Since the automorphism group of $J(n, d)$ acts transitively on $\binom{[n]}{d}$, the isomorphism class of the Terwilliger algebra $\mathcal{T}(x)$ of $J(n, d)$ is independent of the choice of $x$ in $\binom{[n]}{d}$. We will denote $\mathcal{T} := \mathcal{T}(x)$.

In [10], the Terwilliger algebra of the Johnson scheme $J(n, d)$ was determined when $n \ge 3d$. In this paper, we focus on the remaining case, and determine the Terwilliger algebra $\mathcal{T}$ of $J(n, d)$. In Section 2, we introduce intersection matrices and some useful identities. In Section 3, two families of subalgebras $\mathcal{M}^{(n,d)}$ and $\mathcal{N}$ of $\mathrm{Mat}_X(\mathbb{C})$ are constructed. In the last two sections, we show that $\mathcal{T} = \mathcal{M}^{(n,d)}$ when $2d < n < 3d$, and $\mathcal{T} = \mathcal{N}$ when $n = 2d$.

## 2. Intersection matrix

In this section we first introduce some useful identities for intersection matrices, then describe the adjacency matrix of the Johnson scheme $J(n, d)$ in terms of intersection matrices.

Let $V$ be a set of cardinality $v$. Let $H_{i,j}^{r}(v)$ be a binary matrix whose rows and columns are indexed by the elements of $\binom{V}{i}$ and $\binom{V}{j}$ respectively, whose $\alpha_i\alpha_j$-entry is 1 if and only if $|\alpha_i \cap \alpha_j| = r$. We call $H_{i,j}^{r}(v)$ an intersection matrix. For simplicity, write $H_{i,j} := H_{i,j}^{\min(i,j)}$. Now we introduce some useful identities for intersection matrices.

**Lemma 2.1** ([6, *Theorem 3*]). *For* $0 \le l \le \min(i, j)$ *and* $0 \le s \le \min(j, k)$,

$$H_{i,j}^{l}(v)H_{j,k}^{s}(v) = \sum_{g=0}^{\min(i,k)} \left( \sum_{h=0}^{g} \binom{g}{h} \binom{i-g}{l-h} \binom{k-g}{s-h} \binom{v+g-i-k}{j+h-l-s} \right) H_{i,k}^{g}(v).$$

**Lemma 2.2** ([10, *Lemma 4.5*]). *Let* $v$ *be a positive integer.*

(i) *For* $0 \le i \le j \le l \le v$,

$$H_{i,j}(v)H_{j,l}(v) = \binom{l-i}{l-j} H_{i,l}.$$

(ii) *For* $0 \le \max(i, l) \le j \le v$,

$$H_{i,j}(v)H_{j,l}(v) = \sum_{m=0}^{j-\max(i,l)} \binom{v-\max(i,l)-m}{j-\max(i,l)-m} H_{i,l}^{\min(i,l-m)}(v).$$

(iii) *For* $0 \le j \le \min(i, l) \le v$,

$$H_{i,j}(v)H_{j,l}(v) = \sum_{m=0}^{\min(i,l)-j} \binom{\min(i,l)-m}{j} H_{i,l}^{\min(i,l-m)}(v).$$

Pick $x \in \binom{[n]}{d}$. For $0 \le i \le d$, write $\Omega_i := \{y \in \binom{[n]}{d} \mid |x \cap y| = d - i\}$. Then we have the partition $\binom{[n]}{d} = \dot{\bigcup}_{i=0}^{d} \Omega_i$. Now we consider the $m$th adjacency matrix $A_m$ of $J(n, d)$ as a block matrix with respect to this partition. Denote $(A_m)_{|\Omega_i \times \Omega_j}$ the submatrix of $A_m$ with rows indexed by $\Omega_i$ and columns indexed by $\Omega_j$.

In the remainder of this paper, we always assume that $I^{(v,k)}$ denotes the identity matrix of size $\binom{v}{k}$ and $A_m^{(v,k)}$ denotes the $m$th adjacency matrix of $J(v, k)$. In fact $A_m^{(v,k)} = H_{k,k}^{k-m}(v)$.

**Lemma 2.3** ([10, *Lemmas 3.1, 3.5*]). *Let* $A$ *denote the adjacency matrix of* $J(n, d)$. *For* $0 \le i < j \le d$, *we have*

$$A_{|\Omega_i \times \Omega_i} = I^{(d,d-i)} \otimes A^{(n-d,i)} + A^{(d,d-i)} \otimes I^{(n-d,i)},$$
$$A_{|\Omega_i \times \Omega_{i+1}} = H_{d-i,d-i-1}(d) \otimes H_{i,i+1}(n-d),$$
$$A_{|\Omega_i \times \Omega_j} = 0, \quad \text{if } j \ge i + 2,$$

*where "⊗" denotes the Kronecker product of matrices.*

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