Discrete Mathematics 328 (2014) 71–78

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

The crossing number of the Cartesian product of paths with complete graphs

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ARTICLE INFO

Article history:
Received 20 December 2012
Received in revised form 4 March 2014
Accepted 17 March 2014
Available online 18 April 2014

Keywords:
Crossing number
Cartesian product
Zip product
Complete graph

ABSTRACT

In this paper, we determine the crossing number of $K_m \setminus e$ by the construction method for $m \leq 12$ and apply the zip product to obtain that $cr(K_m \Box P_n) = (n-1)cr(K_{m+2} \setminus e) + 2cr(K_{m+1})$ for $n \geq 1$. Furthermore, we have

$$cr(K_m \Box P_n) = \frac{1}{4}\left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{m-2}{2} \right\rfloor \left(n \left\lfloor \frac{m+4}{2} \right\rfloor + \left\lfloor \frac{m-4}{2} \right\rfloor \right)$$

for $n \geq 1$, $1 \leq m \leq 10$, which is consistent with Zheng's conjecture for the crossing number of $K_m \Box P_n$.

1. Introduction

All graphs considered here are simple, finite and undirected and are also connected. For graph theory terminology not defined here, we direct the reader to [7]. A drawing of a graph $G = (V, E)$ is a mapping ϕ that assigns to each vertex in V a distinct point in the plane and to each edge uv in E a continuous arc (i.e., a homeomorphic image of a closed interval) connecting $\phi(u)$ and $\phi(v)$, without passing through the image of any other vertex. In addition, we impose the following conditions on a drawing: (1) no three edges have an interior point in common, (2) if two edges share an interior point p, then they cross at p, and (3) any two edges of a drawing have only a finite number of crossings (common interior points). The crossing number $cr(G)$ of a graph G is the minimum number of edge crossings in any drawing of G. Let D be a drawing of the graph G, and we denote the number of crossings in D by $cr_D(G)$. For more on the theory of crossing numbers, we refer the reader to [8]. The Cartesian product $G \Box H$ of graphs G and H has the vertex set $V(G) \times V(H)$ and the edge set $E(G \Box H) = \{(x_1, y_1)(x_2, y_2) | x_1 = x_2 \text{ and } y_1 y_2 \in E(H) \text{ or } y_1 = y_2 \text{ and } x_1 x_2 \in E(G)\}$.

The investigation of the crossing number of a graph is a classical but very difficult problem (for example, see [8]). In fact, computing the crossing number of a graph is NP-complete [9], and the exact values are known only for very restricted classes of graphs. The crossing numbers of the Cartesian products of graphs have been studied since 1973, when Harary et al. [12] conjectured that $cr(C_m \Box C_n) = (m-2)n$ for $3 \leq m \leq n$. This conjecture has been verified in [1,20–23] for $m \leq 7, n \geq m$. Glebsky and Salazar [10] also showed that the conjecture holds for $n \geq m(m+1)$ and $m \geq 3$. Klešč [15] determined the crossing numbers of the products of all 4-vertex graphs with paths and stars except $cr(K_{1,3} \Box P_n)$, which was earlier determined by Jendroľ and Ščerbová [13], who also obtained $cr(K_{1,3} \Box C_n)$ for $n \geq 1$. In their paper, they

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http://dx.doi.org/10.1016/j.disc.2014.03.011
0012-365X/

conjectured that $cr(S_m\Box P_n) = (n-1)\lfloor \frac{m}{2}\rfloor\lfloor \frac{m-1}{2}\rfloor$ for $m, n \geq 1$. For general n, the conjecture was recently confirmed by Bokal in [5]. Beineke and Ringeisen [3,4] determined the crossing numbers of the products of all 4-vertex graphs with cycles. Klešč [16–18] determined the crossing numbers of the products of all 5-vertex graphs with paths. In particular, he proved that $cr(K_5\Box P_n) = 6n$ for $n \geq 1$ in [16]. Zheng et al. [26] recently proved that $cr(K_6\Box P_n) = 15n+3$ for $n \geq 1$ and

$$cr(K_m\Box P_n) \geq (n-1)cr(K_{m+2}\setminus e) + 2cr(K_{m+1}). \tag{1.1}$$

In their paper, for $n \geq 1$, they conjecture that

$$cr(K_m\Box P_n) = \frac{1}{4}\left\lfloor\frac{m+1}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{m-2}{2}\right\rfloor\left(n\left\lfloor\frac{m+4}{2}\right\rfloor+\left\lfloor\frac{m-4}{2}\right\rfloor\right). \tag{1.2}$$

In this contribution, we show that equality holds in (1.1) for $m \geq 1$ and conjecture that (1.2) holds for $1 \leq m \leq 10$. The approach is seemingly new. To obtain the crossing number of $K_m \setminus e$, we construct a drawing of K_m from the drawing of $K_m \setminus e$ and obtain two lower bound expressions of $cr(K_m \setminus e)$ by the standard counting method used in [14,25]. To prove that equality holds in (1.1), we introduce the zip product operation that was used in [2,5,6,24] and prove a lemma about it with similar sufficient conditions to the ones in [5].

2. Some definitions and lemmas

Definition 2.1. For a graph G, let $A, B \subseteq E(G)$; then, for a drawing ϕ of G, let

$$cr_\phi(A,B) = \sum_{a\in A, b\in B} |\phi(a)\cap\phi(b)|.$$

Additionally, let $cr_\phi(A,A) = cr_\phi(A)$.

Informally, $cr_\phi(A,B)$ denotes the number of crossings between every pair of edges where one edge is in A and the other in B.

For three mutually disjoint subsets $A, B, C \subset E(G)$, the identities

$$cr_\phi(A\cup B) = cr_\phi(A) + cr_\phi(B) + cr_\phi(A,B) \tag{2.1}$$

and

$$cr_\phi(A, B\cup C) = cr_\phi(A,B) + cr_\phi(B,C) \tag{2.2}$$

are noted.

Let G_i, $i = 1, 2$, be a graph with a vertex $v_i \in V(G_i)$ whose neighborhood $N_i = N_{G_i}(v_i)$ has size d. A *zip function* of graphs G_1 and G_2 at vertices v_1 and v_2 is a bijection $\sigma : N_1 \to N_2$. The *zip product* $G_1 \odot_\sigma G_2$ of graphs G_1 and G_2 according to σ is obtained from the disjoint union of $G_1 - v_1$ and $G_2 - v_2$ by adding the edges $u\sigma(u)$, $u \in N_1$.

A drawing D_i of the graph G_i ($i = 1, 2$) defines (up to a circular permutation) a bijection $\pi_i : N_i \to \{1, 2, \ldots, d\}$ that respects the edge rotation around v_i in D_i. The zip function of drawings D_1 and D_2 at vertices v_1 and v_2 is $\sigma : N_1 \to N_2$, $\sigma = \pi_2^{-1}\pi_1$. The zip product $D_1 \odot_\sigma D_2$ of D_1 and D_2 according to σ is obtained from D_1 by adding a mirrored copy of D_2 that has v_2 incident with the infinite face disjointly into some face of D_1 incident with v_1 by removing vertices v_1 and v_2 and small disks around them from the drawings and then joining the edges according to the function σ. For more detail, we refer the reader to [5,6]. For this construction, the following lemmas hold:

Lemma 2.1 ([5]). *For $i = 1, 2$, let D_i be an optimal drawing of G_i, let $v_i \in V(G_i)$ be a vertex of degree d, and let σ be a zip function of D_1 and D_2 at v_1 and v_2. Then, $cr(G_1 \odot_\sigma G_2) \leq cr(G_1) + cr(G_2)$.*

Let $S_n = K_{1,n}$ be a star graph with n vertices of degree 1 (called the leaves of the star) and one vertex of degree n (the center). Let G be a graph and $S \subseteq V(G)$, $|S| = n$. We say that S is k-star-connected in G if there exist k disjoint sets $F_1, F_2, \ldots, F_k \subseteq E(G)$ such that either $G[F_i]$ is a subdivision of S_n with S being the leaves or $G[F_i]$ is a subdivision of S_{n-1} with all its leaves and the center belonging to S.

Lemma 2.2 ([5]). *Let G_1 and G_2 be disjoint graphs, $v_i \in V(G_i)$, $deg(v_i) = d$, and let the neighborhood N_i of v_i be 2-star-connected in $G_i - v_i$, $i = 1, 2$. Then, $cr(G_1 \odot_\sigma G_2) \geq cr(G_1) + cr(G_2)$ for any bijection $\sigma : N_1 \to N_2$.*

Lemma 2.3 ([26]). $cr(K_m \setminus e) \leq \frac{1}{4}\left\lfloor\frac{m+2}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{m-3}{2}\right\rfloor\left\lfloor\frac{m-4}{2}\right\rfloor$.

Some of the proofs in this paper are based on these results for the crossing numbers of complete graphs, more precisely as follows:

Conjecture 2.1 ([11]). $cr(K_m) = \frac{1}{4}\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{m-2}{2}\right\rfloor\left\lfloor\frac{m-3}{2}\right\rfloor$.

It has been proven by Guy [11] for $m \leq 10$ and by Pan and Richter [19] for $m = 11, 12$, respectively.