Discrete Mathematics 328 (2014) 96–101

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

A note on the real part of complex chromatic roots

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ARTICLE INFO

Article history:
Received 27 December 2013
Received in revised form 1 April 2014
Accepted 4 April 2014
Available online 23 April 2014

Keywords:
Chromatic number
Chromatic polynomial
Chromatic roots
Real part

ABSTRACT

A *chromatic root* is a root of the chromatic polynomial of a graph. While the real chromatic roots have been extensively studied and well understood, little is known about the *real parts* of chromatic roots. It is not difficult to see that the largest real chromatic root of a graph with n vertices is $n-1$, and indeed, it is known that the largest real chromatic root of a graph is at most the tree-width of the graph. Analogous to these facts, it was conjectured in Dong et al. (2005) that the real parts of chromatic roots are also bounded above by both $n-1$ and the tree-width of the graph.

In this article we show that for all $k \geq 2$ there exist infinitely many graphs G with tree-width k such that G has non-real chromatic roots z with $\Re(z) > k$. We also discuss the weaker conjecture and prove it for graphs G with $\chi(G) \geq n-3$.

1. Introduction

Let G be a simple graph of order n and size m, and let $\chi(G)$ denote the chromatic number of G. The *chromatic polynomial* $\pi(G, x)$ of G counts the number of proper colourings of the vertices with x colours. If $z \in \mathbb{C}$ satisfies $\pi(G, z) = 0$, then z is called a *chromatic root* of G (the chromatic roots of graphs of order 8 are shown in Fig. 1). A trivial observation is that all of $0, 1, \ldots, \chi(G) - 1$ are chromatic roots—the chromatic number is merely the first positive integer that is *not* a chromatic root. The Four Colour Theorem is equivalent to the fact that 4 is never a chromatic root of a planar graph, and interest in chromatic roots began precisely from this connection. The roots of chromatic polynomials have subsequently received a considerable amount of attention in the literature. Chromatic polynomials also have strong connections to the Potts model partition function studied in theoretical physics, and the complex roots play an important role in statistical mechanics (see, for example, [10]).

A central problem has been to bound the moduli of the chromatic roots in terms of graph parameters. There have been several results regarding this. Brown [2] showed that the chromatic roots of G lie in $|z-1| \leq m-n+1$ and Sokal [10] proved that the chromatic roots lie within $|z| \leq 7.963907\Delta(G)$, where $\Delta(G)$ is the maximum degree of the graph.

Another approach has been to study the *real* chromatic roots of graphs. It is not difficult to see that if r is a real chromatic root of G then $r \leq n-1$ with equality if and only if G is a complete graph. In [4] it was proven that among all real chromatic roots of graphs with order $n \geq 9$, the largest non-integer real chromatic root is $\frac{n-1-\sqrt{(n-3)(n-7)}}{2}$, and extremal graphs were determined. Moreover, Dong et al. [5,6] showed that real chromatic roots are bounded above by $5.664\Delta(G)$ and $\max\{\Delta(G), \lfloor n/3 \rfloor - 1\}$.

The *tree-width* of a graph G is the minimum integer k such that G is a subgraph of a k-tree (given $q \in \mathbb{N}$, the class of *q-trees* is defined recursively as follows: any complete graph K_q is a q-tree, and any q-tree of order $n+1$ is a graph obtained from a

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http://dx.doi.org/10.1016/j.disc.2014.04.007

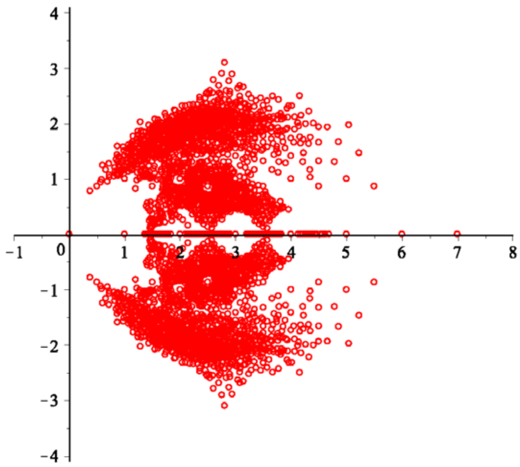

Fig. 1. Chromatic roots of graphs of order 8.

q-tree G of order n, where $n \geq q$, by adding a new vertex and joining it to each vertex of a K_q in G). Thomassen [11] proved that the real chromatic roots are bounded above by the tree-width of the graph.

The problem of finding the largest *real part* of complex chromatic roots seems to be more difficult. In [7] the following conjectures on the real part of complex chromatic roots were proposed.

Conjecture 1.1 ([7, p. 299]). *Let G be a graph with tree-width k. If z is a root of $\pi(G, x)$ then $\Re(z) \leq k$.*

Conjecture 1.2 ([7, p. 299]). *Let G be a graph of order n. If z is a root of $\pi(G, x)$ then $\Re(z) \leq n - 1$.*

It is clear that Conjecture 1.2 is weaker than Conjecture 1.1. In this work, first we present infinitely many counterexamples to Conjecture 1.1 for every $k \geq 2$ (Theorem 2.5). Then, we consider Conjecture 1.2 and prove it for all graphs G with $\chi(G) \geq n - 3$ (Theorem 2.11). (Our numerical computations suggest that graphs which have a large chromatic number are more likely to have chromatic roots whose real parts are close to n.)

2. Main results

A polynomial $f(x)$ in $\mathbb{C}[x]$ is called *Hurwitz quasi-stable* or just *quasi-stable* (resp. *Hurwitz stable* or just *stable*) if every $z \in \mathbb{C}$ such that $f(z) = 0$ satisfies $\Re(z) \leq 0$ (resp. $\Re(z) < 0$). Observe that z is a root of $f(x)$ if and only if $z - c$ is a root of $f(x + c)$, so that every root z of a polynomial $f(x)$ satisfies $\Re(z) \leq c$ (resp. $\Re(z) < c$) if and only if the polynomial $f(x + c)$ is quasi-stable (resp. stable). Thus, bounding the real parts of roots of polynomials is closely related to the Hurwitz stability of polynomials. In the sequel, we will make use of this observation to prove both of our main results.

2.1. Tree-width and the real part of complex chromatic roots

It is not difficult to see that the tree-width of the complete bipartite graph $K_{p,q}$ is equal to $\min(p, q)$, and our counterexamples to Conjecture 1.1 will be these graphs. Note that this conjecture clearly holds for $k = 1$ since the tree-width of a graph is equal to 1 if and only if the graph is a tree. Hence, our counterexamples are for $p \geq 2$.

We shall make use of a particular expansion of the chromatic polynomial. Let G be a graph of order n and size m. Suppose that $\beta : E(G) \to \{1, 2, \ldots, m\}$ is a bijection and C a cycle in G. Let e be the edge of C such that $\beta(e) > \beta(e')$ for any e' in $E(C) - \{e\}$. Then the path $C - e$ is called a *broken cycle* in G with respect to β. Whitney's Broken-Cycle Theorem (see, for example, [7]) states that

$$\pi(G, x) = \sum_{i=1}^{n} (-1)^{n-i} h_i(G) x^i,$$

where $h_i(G)$ is the number of spanning subgraphs of G that have exactly $n - i$ edges and that contain no broken cycles with respect to β.

For two graphs H and G, we denote by $\eta_G(H)$ (resp. $i_G(H)$) the number of subgraphs (respectively induced subgraphs) of G which are isomorphic to H. For example, for the graph H in Fig. 2, we have $\eta_H(K_3) = i_H(K_3) = 2$, $\eta_H(2K_2) = 8$ and $i_H(2K_2) = 0$. The following result gives formulas for the first few coefficients of the chromatic polynomial by counting certain (induced) subgraphs of the graph.