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Properties of mixed Moore graphs of directed degree one

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ABSTRACT

Mixed graphs of order n such that for any pair of vertices there is a unique trail of length at most k between them are known as mixed Moore graphs. These extremal graphs may only exist for diameter $k = 2$ and certain (infinitely many) values of n. In this paper we give some properties of mixed Moore graphs of directed degree one and reduce their existence to the existence of some (undirected) strongly regular graphs.

1. Introduction

Notation and terminology

A *mixed graph* G may contain (undirected) *edges* as well as directed edges (also known as *arcs*). From this point of view, a *graph* [resp. *directed graph* or *digraph*] has all its edges undirected [resp. directed]. The *undirected degree* of a vertex v, denoted by $d(v)$ is the number of edges incident to v. The *out-degree* [resp. *in-degree*] of v, denoted by $d^+(v)$ [resp. $d^-(v)$], is the number of arcs emanating from [resp. to] v. If $d^+(v) = d^-(v) = z$ and $d(v) = r$, for all $v \in V$, then G is said to be *totally regular* of degree d, where $d = r + z$. A *trail* from u to v of length $l \geq 0$ is a sequence of $l + 1$ vertices, $u_0u_1 \dots u_{k-1}u_l$, such that $u = u_0$, $v = u_l$ and each pair $u_{i-1}u_i$ is either an edge or an arc of G. A trail whose vertices are all different is called a *path*. A *directed path* is a path containing only arcs. The length of a shortest path from u to v is the *distance* from u to v, denoted by $\text{dist}(u, v)$. Note that $\text{dist}(u, v)$ may be different from $\text{dist}(v, u)$ when shortest paths between u and v involve arcs. The maximum distance between pairs of vertices of G, denoted by $\text{diam}(G)$, is called the *diameter* of G. A *directed cycle* [resp. *undirected cycle*] of length l is a trail of length l involving only arcs [resp. edges] whose vertices are all different except $u = v$. A *digon* is a directed cycle of length two. A *triangle* is an undirected cycle of length three.

Mixed Moore graphs

In 1978 Bosák studied those mixed graphs such that for any pair of vertices there is a unique trail of length at most the diameter between them. In some sense, Bosák generalized the concepts of Moore graph and Moore digraph introduced some years before, by allowing the existence of both edges and arcs simultaneously into the topology, giving the structure of a mixed graph.

The maximum number of vertices n for a mixed graph G of diameter k, maximum undirected degree r and maximum out-degree z is bounded by

$$n \leq M_{r,z,k} = 1 + (r + z) + [z(r + z) + r(r + z - 1)] + \cdots + [z(r + z)^{k-1} + r(r + z - 1)^{k-1}] \quad (1)$$

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Table 1
Pairs (r, z) satisfying Eq. (3) and such that $M_{r,z,2} \leq 200$.

$M_{r,z,2}$	r	z	d	Existence	Uniqueness
6	1	1	2	$Ka(2, 2)$	Yes
12	1	2	3	$Ka(3, 2)$	Yes
18	3	1	4	Bosák graph	Yes
20	1	3	4	$Ka(4, 2)$	Yes
30	1	4	5	$Ka(5, 2)$	Yes
40	3	3	6	Unknown	Unknown
42	1	5	6	$Ka(6, 2)$	Yes
54	3	4	7	Unknown	Unknown
56	1	6	7	$Ka(7, 2)$	Yes
72	1	7	8	$Ka(8, 2)$	Yes
84	7	2	9	Unknown	Unknown
88	3	6	9	Unknown	Unknown
90	1	8	9	$Ka(9, 2)$	Yes
108	3	7	10	Jørgensen graphs	No
110	1	9	10	$Ka(10, 2)$	Yes
132	1	10	11	$Ka(11, 2)$	Yes
150	7	5	12	Unknown	Unknown
154	3	9	12	Unknown	Unknown
156	1	11	12	$Ka(12, 2)$	Yes
180	3	10	13	Unknown	Unknown
182	1	12	13	$Ka(13, 2)$	Yes

as it can be seen in [8]. This bound is known as *the mixed Moore bound* and mixed graphs attaining it are called *mixed Moore graphs*. Such extremal mixed graphs are totally regular of degree $d = r + z$ and they have the property that for any ordered pair (u, v) of vertices there is a unique trail of length at most the (finite) diameter k between them.

Mixed Moore graphs are known as *Moore graphs* when they contain only edges. These (undirected) graphs are well studied: in the case of diameter $k = 2$, Hoffman and Singleton [6] proved existence and unicity for $r = 2, 3, 7$ and possibly $r = 57$, and nonexistence for other degrees. They also showed that for diameter $k = 3$ and degree $r > 2$ Moore graphs do not exist. The enumeration of Moore graphs of diameter $k > 3$ was concluded by Damerell [4], using the theory of distance-regularity to prove nonexistence unless $r = 2$ (an independent proof was given by Bannai and Ito [1]). On the other hand, mixed Moore graphs are known as *Moore digraphs* when they contain only arcs. It is well known that Moore digraphs do only exist in the trivial cases, $z = 1$ or $k = 1$, which correspond to the directed cycle of order $k+1$ and the complete digraph of order $z + 1$ respectively (see [10,3]).

Mixed Moore graphs containing both edges and arcs are called *proper mixed Moore graphs*. Nguyen, Miller and Gimbert [9] proved the nonexistence of proper mixed Moore graphs of diameter $k \geq 3$. But some mixed Moore graphs of diameter $k = 2$ are known to exist. If A is the adjacency matrix of a mixed Moore graph of diameter $k = 2$ and undirected degree r, due to the uniqueness of trails of length at most two between any pair of vertices, A satisfies

$$I + A + A^2 = J + rI \tag{2}$$

where I and J denote the identity and the all-ones matrix, respectively. By (1) and (2) one can compute the spectrum of A in terms of the undirected degree r and the directed degree z. Since the main diagonal of A is all 0's, the sum of all the eigenvalues of A is equal to zero. This imposes a necessary condition for the existence of a mixed Moore graph of diameter two, as it appears in [2]:

Theorem 1 ([2]). *Let G be a (proper) mixed graph of diameter two. Then, G is totally regular with directed degree $z \geq 1$ and undirected degree $r \geq 1$. Moreover, there must exist a positive odd integer c such that*

$$r = \frac{1}{4}(c^2 + 3) \quad \textit{and} \quad c|(4z - 3)(4z + 5). \tag{3}$$

As an example, for $z = 1$ the only feasible values of r are 1, 3 and 21, corresponding to the possible values of c, namely, $c = 1, 3$ and 9. The vertices of the *Kautz digraph* $Ka(d, k)$, $d \geq 2$, $k \geq 1$, are words of length k on an alphabet S of $d + 1$ letters without two consecutive identical letters. There is an arc from vertex $(v_0, v_1, \ldots, v_{k-1})$ to vertices $(v_1, \ldots, v_{k-1}, x)$, where $x \in S \setminus \{v_{k-1}\}$. It is known that $Ka(d, k)$ has order $(d + 1)d^{k-1}$ and diameter k. Kautz digraphs $Ka(d, 2)$ are proper mixed Moore graphs for every $z \geq 2$ and $r = 1$ (if every digon is replaced by an edge) where $d = r + z$, as noted in [9]. Another interesting proper mixed Moore graph of order 18 (see Fig. 3) was given by Bosák in [2] (and its uniqueness was proved later in [9]). Recently, Jørgensen [7] found two non-isomorphic proper mixed Moore graphs of order 108. Besides, there are many pairs (r, z) satisfying Eq. (3) for which the existence of a proper mixed Moore graph is not yet known (see Table 1).