Discrete Mathematics 338 (2015) 615–620

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

On characterizing radio k-coloring problem by path covering problem

Ushnish Sarkar [a,b], Avishek Adhikari [b,*]

[a] S. K. B. Polytechnic, Keshiary, W. B., India

[b] Department of Pure Mathematics, University of Calcutta, India

ARTICLE INFO

Article history:
Received 22 July 2014
Received in revised form 19 November 2014
Accepted 25 November 2014
Available online 24 December 2014

Keywords:
Radio k-coloring
Hamiltonian path
Path covering number
Circulant graph

ABSTRACT

Let G be a finite simple graph. For an integer $k \geq 1$, a radio k-coloring of G is an assignment f of non-negative integers to the vertices of G satisfying the condition $|f(u) - f(v)| \geq k + 1 - d(u, v)$ for any two distinct vertices u, v of G. The span of f is the largest integer assigned to a vertex of G by f and radio k-chromatic number of G, denoted by $rc_k(G)$, is the minimum span over all radio k-colorings of G. For $k = 2$, the radio k-coloring becomes $L(2, 1)$ coloring problem. On the other hand, path covering problem deals with finding minimum number of vertex disjoint paths required to exhaust all the vertices of G. Georges et al. (1994) explored an elegant relation between $L(2, 1)$-coloring problem and path covering problem. As an extension of their work, we characterize the radio k-coloring problem for any $k \geq 2$ of a graph G by the path covering problem of G^c, where either G is triangle free or there is a Hamiltonian path in each component of G^c. As an application, for any such graph, if the exact value or an upper bound is known for any $rc_p(G)$, $p \geq 2$, we can get the exact value or an upper bound of $rc_k(G)$ for all $k \geq 2$. Determination of radio k-chromatic numbers of complete multi-partite graphs, a certain family of circulant graphs and join of circulant graphs of a certain family are among some other applications.

1. Introduction

Many graph coloring problems stem from a problem widely known as the frequency assignment problem (FAP) in communication network. In FAP, frequencies (non-negative integers) are assigned to the transmitters in a wireless network in an economic way. But as the proximity of transmitters increases, the mutual differences among the frequencies allotted to them should be greater to avoid interference. So the task is to minimize the span, i.e., the maximum frequency assigned, while satisfying the interference constraints. Hale [11] modelled this as a graph coloring problem. Roberts [24] proposed a variation of this problem taking a cue from which Griggs and Yeh [10] introduced $L(p_1, p_2, \ldots, p_m)$-coloring of a simple graph $G = (V, E)$ which is a function $f : V \to \mathbb{N}$ such that $|f(u) - f(v)| \geq p_i$ when $d(u, v) = i$, for $i = 1, 2, \ldots, m$, where $\mathbb{N}$ is the set of all non-negative integers. Interestingly, for a simple finite graph G and for any $k \geq 1$, if $p_i = k - i + 1,\ 1 \leq i \leq m$, the problem becomes the radio k-coloring problem introduced by Chartrand et al. [4,6] which finds motivation in FM channel assignments. In other words, if $\mathbb{N}$ is the set of all non-negative integers, then for any positive integer $k \geq 1$, a radio k-coloring f of a finite simple graph G is a mapping $f : V \to \mathbb{N}$ such that for any two vertices u, v in G,

$$|f(u) - f(v)| \geq k + 1 - d(u, v). \tag{1}$$

* Corresponding author.
E-mail addresses: usn.prl@gmail.com (U. Sarkar), avishek.adh@gmail.com (A. Adhikari).

http://dx.doi.org/10.1016/j.disc.2014.11.014
0012-365X/

The span of a radio k-coloring f, denoted by *span(f)*, is $(\max_{v\in V} f(v) - \min_{v\in V} f(v))$ and the *radio k-chromatic number* of G, denoted by $rc_k(G)$ is defined as $\min_f\{span(f)\ :\ f \text{ is a radio } k\text{-coloring of } G\}$. Without loss of generality, we shall assume $\min_{v\in V} f(v) = 0$ for any radio k-coloring f on G. Any radio k-coloring f on G with span $rc_k(G)$ is referred as $rc_k(G)$-coloring or simply rc_k-coloring (when there is no confusion regarding the underlying graph).

So far radio k-coloring of graphs has been studied for $k \geq 2\cdot diam(G)-2$, $k = diam(G)$, $k = diam(G)-1$, $k = diam(G)-2$, $k = 3$, $k = 2$. For $k = diam(G)$, $k = diam(G) - 1$ and $k = diam(G) - 2$, the radio k-coloring is referred as *radio coloring*, *antipodal coloring* and *near-antipodal coloring* respectively while the corresponding radio k-chromatic numbers are known as *radio number*, *antipodal number* and *near-antipodal number* of G respectively. When $k = 2$, the problem reduces to the $L(2, 1)$-coloring problem introduced by Griggs and Yeh [10]. Note that $rc_2(G)$ is sometimes denoted as $\lambda_{2,1}(G)$ or $\lambda(G)$.

On the other hand, a *path covering* of a graph G is a set of vertex disjoint paths through all the vertices of G. A path covering of minimum cardinality of G is called a *minimum path covering* of G and its size is called the *path covering number* of G, denoted by $c(G)$. Finding a minimum path covering has applications in establishing ring protocols, codes optimization and mapping parallel programs to parallel architectures [1,2,22,23].

2. Previous works

So far, the radio k-chromatic numbers are known for very few families of graphs for specified values of k. Chartrand et al. [4,5] studied the radio numbers of paths and cycles while Liu and Zhu [21] obtained their exact values. The radio k-chromatic number of path P_n has been obtained in [21,13,15] for $k = n-1$, $n-2$, $n-3$ respectively. Kola and Panigrahi [16] have determined $rc_{n-4}(P_n)$ for an odd integer n. Liu generalized the results for paths to spider, i.e., trees with at most one vertex of degree greater than two, and obtained exact radio numbers in some specific cases [18]. Li et al. determined the radio number of a complete m-ary tree in [17]. Khennoufa et al. in [14] determined the radio number and the antipodal number of any hypercube by using generalized binary Gray codes. Moreover for hypercubes, upper bounds and lower bounds for radio k-chromatic numbers when $k \geq 2$ and their exact values when $k \geq 2\cdot$diameter-2 were obtained in [12]. Liu et al. in [19,20] studied radio numbers of squares of cycles and paths respectively. For powers of cycles i.e. C_n^r, Saha et al. obtained antipodal numbers for some values of n and r and bounds for the remaining cases [25].

It may be perceived that finding radio k-chromatic numbers, for any $k \geq 2$, even for paths, cycles and their powers is a challenging task. Since almost all graphs are asymmetric (i.e. its automorphism group is the identity group) [9], finding radio k-chromatic number for general graphs for any $k \geq 2$ is arguably a much more difficult job. Even in the literature, up to the best of our knowledge, there is no theoretical upper bound of $rc_k(G)$ for any graph G and for any $k \geq 3$. For any graph G, it was proved in [3] that $rc_2(G) \leq \Delta^2 + \Delta$ and conjectured in [10] that $rc_2(G) \leq \Delta^2$, Δ being the maximum degree in G.

3. Our contributions

In [8], Georges et al. investigated the relationship between $rc_2(G)$ and $c(G^c)$, where G^c is the complement of the graph G, and established the following beautiful results. In that paper, $rc_2(G)$ was denoted by $\lambda(G)$.

Theorem 3.1 (*[8]*). *Let G be a graph with n vertices.*
(i) $\lambda(G) \leq (n-1)$ *if and only if* $c(G^c) = 1$.
(ii) *Let r be an integer,* $r \geq 2$. *Then* $\lambda(G) = n + r - 2$ *if and only if* $c(G^c) = r$.

Let $\mathcal{G}^*$ be the collection of all finite simple graphs. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two families of graphs defined by $\mathcal{G}_1 = \{G \in \mathcal{G}^* \mid G \text{ is triangle free}\}$ and $\mathcal{G}_2 = \{G \in \mathcal{G}^* \mid \text{each component in } G^c \text{ has a Hamiltonian path}\}$. In this paper we extend the above result for any $k \geq 2$ and for any graph which is either in $\mathcal{G}_1$ or in $\mathcal{G}_2$. In fact we have obtained an upper bound for $rc_k(G)$ in terms of $c(G^c)$ for any graph G, where G^c is the complement of the graph G. For a graph $G \in \mathcal{G}_1 \cup \mathcal{G}_2$ and for $k \geq 2$, we show this upper bound to be a characterization for the existence of a Hamiltonian path in G^c and otherwise, i.e., if there is no Hamiltonian path in G^c, we then obtain a closed formula for $rc_k(G)$. Consequently for any graph G in $\mathcal{G}_1 \cup \mathcal{G}_2$, if the exact value or an upper bound is known for any $rc_p(G)$, $p \geq 2$, we can get the exact value or an upper bound of $rc_k(G)$ for all $k \geq 2$. Other applications include determining radio k-chromatic numbers of complete multi-partite graphs, certain family of circulant graphs and join of circulant graphs of a certain family.

4. Preliminaries

Throughout this paper, unless otherwise stated, graphs are taken as finite and simple with at least two vertices. Let L be a rc_k-coloring on a graph $G = (V, E)$. An integer $i \in \{0, 1, \ldots, rc_k(G)\}$ is a hole in L if i is not assigned to any vertex of G by L. Let $L_i^k(G) = \{v \in V | L(v) = i\}$ and $l_i^k(G) = |L_i^k(G)|$. We replace $L_i^k(G)$ by L_i and $l_i^k(G)$ by l_i if there is no confusion regarding G and k. For a fixed k, the vertices of L_i^k are represented by $v_j^{i(k)}$ (or v_j^i when there is no confusion regarding k), $1 \leq j \leq l_i$, and if $l_i = 1$, we replace $v_j^{i(k)}$ by $v^{i(k)}$ (or simply v_j^i by v^i if there is no confusion regarding k). In a rc_k-coloring L on G, a color i is referred as a multiple color if $l_i \geq 2$.

For definitions of Hamiltonian path, connectivity and independence number of a graph and disjoint union and join of two graphs, the reader is referred to [26]. Note that connectivity and independence number of a graph G and disjoint union and join of two graphs G, H are denoted by $\kappa(G)$, $\alpha(G)$, $G + H$ and $G \vee H$ respectively. The reader is further referred to [9]