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Minimum order of graphs with given coloring parameters

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ABSTRACT

A complete k-coloring of a graph $G = (V, E)$ is an assignment $\varphi : V \to \{1, \dots, k\}$ of colors to the vertices such that no two vertices of the same color are adjacent, and the union of any two color classes contains at least one edge. Three extensively investigated graph invariants related to complete colorings are the minimum and maximum number of colors in a complete coloring (*chromatic number* $\chi(G)$ and *achromatic number* $\psi(G)$, respectively), and the *Grundy number* $\Gamma(G)$ defined as the largest k admitting a complete coloring φ with exactly k colors such that every vertex $v \in V$ of color $\varphi(v)$ has a neighbor of color i for all $1 \le i < \varphi(v)$. The inequality chain $\chi(G) \le \Gamma(G) \le \psi(G)$ obviously holds for all graphs G. A triple (f, g, h) of positive integers at least 2 is called *realizable* if there exists a *connected* graph G with $\chi(G) = f$, $\Gamma(G) = g$, and $\psi(G) = h$. In Chartrand et al. (2010), the list of realizable triples has been found. In this paper we determine the minimum number of vertices in a connected graph with chromatic number f, Grundy number g, and achromatic number h, for all realizable triples (f, g, h) of integers. Furthermore, for $f = g = 3$ we describe the (two) extremal graphs for each $h \ge 6$. For $h \in \{4, 5\}$, there are more extremal graphs, their description is given as well.

1. Introduction

A *complete coloring* of a graph is an assignment of colors to the vertices in such a way that adjacent vertices receive different colors, and there is at least one edge between any two color classes. In other words, the coloring is proper and the number of colors cannot be decreased by identifying two colors.

Let $G = (V, E)$ be any simple undirected graph. The minimum number of colors in a *proper* coloring is the *chromatic number* $\chi(G)$, and all proper χ-colorings are necessarily complete. The maximum number of colors in a *complete* coloring is the *achromatic number* $\psi(G)$. Every graph admits a complete coloring with exactly k colors for all $\chi \le k \le \psi$ (Harary et al. [15]). An important variant of complete coloring, called *Grundy coloring* or Grundy numbering, requires a proper coloring $\varphi : V \to \{1, \dots, k\}$ such that every vertex $v \in V$ has a neighbor of color i for each $1 \le i < \varphi(v)$. The largest integer k for which there exists a Grundy coloring of G is denoted by $\Gamma(G)$ and is called the *Grundy number* of G. Certainly,

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$\Gamma(G)$ is sandwiched between $\chi(G)$ and $\psi(G)$. One should emphasize that $\chi(G)$ and $\psi(G)$ are defined in terms of unordered colorings, i.e., permutation of colors does not change the required property of a coloring. On the other hand, in a Grundy coloring the order of colors is significant.

Proper colorings have found a huge amount of applications and hence, besides their high importance in graph theory, they are very well motivated from the practical side, too. The chromatic number occurs in lots of optimization problems. The achromatic number looks less practically motivated, nevertheless it expresses the worst case of a coloring algorithm which creates a proper color partition of a graph in an arbitrary way and then applies the improvement heuristic of identifying two colors as long as no monochromatic edge is created. Grundy colorings have strong motivation from game theory; moreover, $\Gamma(G)$ describes the worst case of First-Fit coloring algorithm when applied to a graph G if we do not know the graph in advance, the vertices arrive one by one, and we irrevocably assign the smallest feasible color to each new vertex as a best local choice. Then the number of colors required for a worst input order is exactly $\Gamma(G)$. For this reason, $\Gamma(G)$ is also called the *on-line First-Fit chromatic number* of G in the literature. An overview of on-line colorings and a detailed analysis of the First-Fit version is given in [4]. A more extensive survey on the subject can be found in [19]. The performance of First-Fit is much better on the average than in the worst case. This is a good reason that it has numerous successful applications. This nicely shows from a practical point of view that the Grundy number is worth investigating.

The definition of Grundy number is usually attributed to Christen and Selkow [10], although its roots date back to the works of Grundy [13] four decades earlier; and in fact $\Gamma(G)$ of an undirected graph G is equal to that of the digraph in which each edge of G is replaced with two oppositely oriented arcs. In general, computing the Grundy number is NP-hard, and it remains so even when restricted to some very particular graph classes, e.g., to bipartite graphs or complements of bipartite graphs ([16,25], respectively). Actually, the situation is even worse: there does not exist any polynomial-time approximation scheme to estimate $\Gamma(G)$ unless $\mathsf{P} = \mathsf{NP}$ [20], and for every integer c it is coNP-complete to decide whether $\Gamma(G) \le c\,\chi(G)$, and also whether $\Gamma(G) \le c\,\omega(G)$, where $\omega(G)$ denotes the clique number of G (see [1]). Several bounds on $\Gamma(G)$ in terms of other graph invariants were given, e.g., in [5,26,27]. On the other hand, by the finite basis theorem of Gyárfás et al. [14] the problem of deciding whether $\Gamma(G) \ge k$ can be solved in polynomial time, when k is a fixed integer (see also [6] for results on Grundy critical graphs). Moreover, there are known efficient algorithms to determine the Grundy number of trees [17] and more generally of partial k-trees [23].

Concerning the achromatic number, on the positive side there exists a constant-approximation for trees [9] and a polynomial-time exact algorithm for complements of trees [24]. But in a sense, the computation of $\psi(G)$ is harder than that of $\Gamma(G)$. It is NP-complete to determine $\psi(G)$ on connected graphs that are simultaneously interval graphs and co-graphs [3], and even on trees [7,11]. Moreover, no randomized polynomial-time algorithm can generate with high probability a complete coloring with $C\psi(G)/\sqrt{n}$ colors for arbitrarily large constant C, unless $\mathsf{NP} \subseteq \mathsf{RTime}(n^{\mathrm{poly}\log n})$, and under the same assumption $\psi(G)$ cannot be approximated deterministically within a multiplicative $\lg^{1/4-\varepsilon} n$, for any $\varepsilon > 0$ [22], although some $o(n)$-approximations are known [9,21].

The strong negative results above concerning algorithmic complexity also mean a natural limitation on structural dependencies, for all the three graph invariants χ, Γ, ψ. On the other hand, quantitatively, the triple (χ, Γ, ψ) can take any non-decreasing sequence of integers at least 2. (The analogous assertion for (χ, ψ) without Γ appeared in [2].) For example, if $\chi = \Gamma = 2$, then properly choosing the size of a union of complete graphs on two vertices will do for any given ψ. Assuming connectivity, however, makes a difference. Let us call a triple (f, g, h) of integers with $2 \le f \le g \le h$ *realizable* if there exists a *connected* graph G such that $\chi(G) = f$, $\Gamma(G) = g$, and $\psi(G) = h$. It was proved by Chartrand et al. [8] that a triple is realizable if and only if either $g \ge 3$ or $f = g = h = 2$.

Here we address the naturally arising question of smallest connected graphs with the given coloring parameters. Namely, for a realizable triple (f, g, h), let us denote by $n(f, g, h)$ the minimum order of a connected graph G with $\chi(G) = f$, $\Gamma(G) = g$ and $\psi(G) = h$. The lower bound $n(f, g, h) \ge 2h - f$ was proved in [8, Theorem 2.10] in the stronger form $2\psi - \omega$ (where ω denotes clique number), and this estimate was also shown to be tight for $g = h$. On the other hand the order of graphs constructed there to verify that the triple (f, g, h) is realizable was rather large, and had a high growth rate. In particular, for every fixed f and g, the number of vertices in the graphs of [8] realizing (f, g, h) grows with h^2 as h gets large, while the lower bound is linear in h. For instance, the construction for $f = g$, described in [18], takes the complete graph K_f together with a pendant path P_k, having properly chosen number of vertices k, and applies the facts that very long paths make ψ arbitrarily large and that the removal of the endvertex of a path (or actually any vertex of any graph) decreases ψ by at most 1, as proved in [12].

In this paper we determine the exact value of $n(f, g, h)$ for every realizable triple (f, g, h), showing that the lower bound $2h - f$ is either tight or just one below optimum. It is easy to see that the complete graph K_f verifies $n(f, f, f) = f$ for all $f \ge 2$. For the other cases it will turn out that the formula depends on whether $f < g$. These facts are summarized in the following two theorems; the case $g = h$ was already discussed in [8].

Theorem 1. *For $2 \le f < g$ and for $f = g = h$, $n(f, g, h) = 2h - f$.*

Theorem 2. *For $2 < f = g < h$, $n(f, g, h) = 2h - f + 1$.*

Remark 1. As one can see, the minimum does not depend on g, apart from the distinction between $f = g$ and $f < g$.