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Planar graphs with girth at least 5 are (3, 5)-colorable

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ABSTRACT

A graph is $(d_1, \ldots, d_r)$-colorable if its vertex set can be partitioned into r sets $V_1, \ldots, V_r$ where the maximum degree of the graph induced by V_i is at most d_i for each $i \in \{1, \ldots, r\}$. Let $\mathcal{G}_g$ denote the class of planar graphs with minimum cycle length at least g. We focus on graphs in $\mathcal{G}_5$ since for any d_1 and d_2, Montassier and Ochem constructed graphs in $\mathcal{G}_4$ that are not (d_1, d_2)-colorable. It is known that graphs in $\mathcal{G}_5$ are (2, 6)-colorable and (4, 4)-colorable, but not all of them are (3, 1)-colorable. We prove that graphs in $\mathcal{G}_5$ are (3, 5)-colorable, leaving two interesting questions open: (1) are graphs in $\mathcal{G}_5$ also $(3, d_2)$-colorable for some $d_2 \in \{2, 3, 4\}$? (2) are graphs in $\mathcal{G}_5$ indeed (d_1, d_2)-colorable for all $d_1 + d_2 \geq 8$ where $d_2 \geq d_1 \geq 1$?

1. Introduction

Let $[n] = \{1, \ldots, n\}$. Only finite, simple graphs are considered. Given a graph G, let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. A *neighbor* of a vertex v is a vertex adjacent to v, and let $N(v)$ denote the set of neighbors of v. The *degree* of v, denoted by $d(v)$, is $|N(v)|$. The *degree* of a face f, denoted by $d(f)$, is the length of a shortest boundary walk of f. A *k-vertex*, *k^+-vertex*, and *k^--vertex* are vertices of degree k, at least k, and at most k, respectively. A *k-face*, *k^+-face* is a face of degree k, at least k, respectively. The *girth* of a graph is the length of a shortest cycle.

A graph is $(d_1, \ldots, d_r)$-colorable if its vertex set can be partitioned into r sets $V_1, \ldots, V_r$ where the maximum degree of the graph induced by V_i is at most d_i for each $i \in [r]$; in other words, there exists a function $f : V(G) \to [r]$ where the graph induced by vertices of color i has maximum degree at most d_i for $i \in [r]$.

There are many papers that study $(d_1, \ldots, d_r)$-colorings of sparse graphs resulting in corollaries regarding planar graphs, sometimes with restrictions on the length of a smallest cycle. The well-known four color theorem [1,2] is exactly the statement that planar graphs are (0, 0, 0, 0)-colorable. Cowen, Cowen, and Woodall [7] proved that planar graphs are (2, 2, 2)-colorable, and Eaton and Hull [8] and Škrekovski [11] proved that this is sharp by exhibiting non-$(1, k, k)$-colorable planar graphs for each k. Thus, the problem is completely solved when $r \geq 3$.

Let $\mathcal{G}_g$ denote the class of planar graphs with girth at least g. Given any d_1 and d_2, consider the following graph constructed by Montassier and Ochem [10]. Let $X_i(d_1, d_2)$ be a copy of K_{2,d_1+d_2+1} where one part is $\{x_i, y_i\}$. Obtain $Y(d_1, d_2)$ in the following way: start with $X_1(d_1, d_2), \ldots, X_{d_1+2}(d_1, d_2)$ and identify $x_1, \ldots, x_{d_1+2}$ into x, and add the edges $y_1y_2, \ldots, y_1y_{d_1+2}$. It is easy to verify that $Y(d_1, d_2)$ is in $\mathcal{G}_4$ but it is not (d_1, d_2)-colorable.

Therefore, we focus on graphs in $\mathcal{G}_5$. There are also many papers [3,5,9,6,4,10] that investigate (d_1, d_2)-colorability for graphs in $\mathcal{G}_g$ for $g \geq 6$; see [10] for the rich history. For example, Borodin, Ivanova, Montassier, Ochem, and Raspaud [3]

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constructed a graph in $\mathcal{G}_6$ (and thus also in $\mathcal{G}_5$) that is not $(0, k)$-colorable for any k. The question of determining if there exists a finite k where all graphs in $\mathcal{G}_5$ are $(1, k)$-colorable is not yet known and was explicitly asked in [10]. On the other hand, Borodin and Kostochka [5] and Havet and Sereni [9], respectively, proved results that imply graphs in $\mathcal{G}_5$ are $(2, 6)$-colorable and $(4, 4)$-colorable.

In this paper, we prove the following theorem, which is not implied by the aforementioned results.

Theorem 1.1. *Planar graphs with girth at least 5 are $(3, 5)$-colorable.*

This solves one of the previously unknown cases of the following question.

Question 1.2. *Are planar graphs with girth at least 5 indeed (d_1, d_2)-colorable for all $d_1 + d_2 \geq 8$ where $d_2 \geq d_1 \geq 1$?*

The only remaining case of Question 1.2 is when $d_1 = 1$ and $d_2 = 7$. As mentioned before, interestingly enough, we do not know even if there is a finite k where graphs in $\mathcal{G}_5$ are $(1, k)$-colorable.

Since there are non-$(3, 1)$-colorable graphs in $\mathcal{G}_5$ [10], Theorem 1.1 implies that the minimum d where graphs in $\mathcal{G}_5$ are $(3, d)$-colorable is in $\{2, 3, 4, 5\}$; determining this d would be interesting.

In the figures throughout this paper, the white vertices do not have incident edges besides the ones drawn, and the black vertices may have other incident edges.

In Section 2, we prove structural lemmas for non-(d_1, d_2)-colorable graphs with minimum order. In Section 3, we reveal some more structures of minimum counterexamples to Theorem 1.1 by focusing on the case when $d_1 = 3$ and $d_2 = 5$. Finally, we prove Theorem 1.1 by using a discharging procedure in Section 4.

2. Non-(d_1, d_2)-colorable graphs with minimum order

In this section, we prove structural lemmas regarding non-(d_1, d_2)-colorable graphs with minimum order; let $H(d_1, d_2)$ be such a graph. It is easy to see that the minimum degree of (a vertex of) $H(d_1, d_2)$ is at least 2 and $H(d_1, d_2)$ is connected.

Given a (partial) coloring f of $H(d_1, d_2)$ and $i \in [2]$, a vertex v with $f(v) = i$ is *i-saturated* if v is adjacent to d_i neighbors colored i. By definition, an i-saturated vertex has at least d_i neighbors.

Lemma 2.1. *Let $H = H(d_1, d_2)$ where $d_1 \leq d_2$. If v is a 2-vertex of H, then v is adjacent to two $(d_1+2)^+$-vertices, one of which is a $(d_2 + 2)^+$-vertex.*

Proof. Let $N(v) = \{v_1, v_2\}$ and let f be a coloring of $H - v$ obtained by the minimality of H. If $f(v_1) = f(v_2)$, then letting $f(v) \in [2] \setminus \{f(v_1)\}$ gives a coloring of H, which is a contradiction. Without loss of generality, assume that $f(v_1) = 1$ and $f(v_2) = 2$. Since setting $f(v) = 1$ must not give a coloring of H, we know v_1 is 1-saturated. Since setting $f(v_1) = 2$ and $f(v) = 1$ must not give a coloring of H, we know v_1 has a neighbor colored 2. This implies $d(v_1) \geq d_1 + 2$. Similar logic implies that $d(v_2) \geq d_2 + 2$. □

Lemma 2.2. *Let $H = H(d_1, d_2)$ where $2 \leq d_1 \leq d_2$. If v is a 3-vertex of H, then v is adjacent to at least two $(d_1+2)^+$-vertices, one of which is a $(d_2 + 2)^+$-vertex.*

Proof. Let $N(v) = \{v_0, v_1, v_2\}$ and let f be a coloring of $H - v$ obtained by the minimality of H. If $f(v_0) = f(v_1) = f(v_2)$, then letting $f(v) \in [2] \setminus \{f(v_0)\}$ gives a coloring of H, which is a contradiction. Without loss of generality, assume that $f(v_1) = 1$ and $f(v_2) = 2$. Further assume that $f(v_0) = i$ for some $i \in [2]$ and let $j \in [2] \setminus \{i\}$.

Since setting $f(v) = j$ must not give a coloring of H, we know that v_j is j-saturated. Since setting $f(v) = j$ and $f(v_j) = i$ must not give a coloring of H, we know that v_j has a neighbor colored i. This implies $d(v_j) \geq d_j + 2$. Since setting $f(v) = i$ must not give a coloring of H, we know either v_0 or v_i is i-saturated. If both $d(v_0), d(v_i) \leq d_i+1$, then recolor each i-saturated vertex in $\{v_0, v_i\}$ with color j, and set $f(v) = i$ to obtain a coloring of H, which is a contradiction. Therefore either v_0 or v_i has degree at least $d_i + 2$. □

Lemma 2.3. *Let $H = H(d_1, d_2)$ where $d_1 + 1 \leq d_2$. If v is a $(d_1 + d_2 + 1)^-$-vertex of H, then v is adjacent to at least one $(d_1 + 2)^+$-vertex.*

Proof. Suppose that no neighbor of v is a $(d_1 + 2)^+$-vertex and let f be a coloring of $H - v$ obtained by the minimality of H. Both colors 1 and 2 must appear on $N(v)$; otherwise, we can easily obtain a coloring of H, which is a contradiction. Since setting $f(v) = 2$ must not give a coloring of H and v cannot be adjacent to a 2-saturated vertex (since a 2-saturated neighbor of v has degree at least $d_2 + 1 \geq d_1 + 2$), we know that v has at least $d_2 + 1$ neighbors colored 2. Since setting $f(v) = 1$ must not give a coloring of H, we know that either v has at least $d_1 + 1$ neighbors colored 1 or v has a 1-saturated neighbor. The former case is impossible because $d(v) \leq d_1 + d_2 + 1$. Since each neighbor of v is a $(d_1 + 1)^-$-vertex, each 1-saturated neighbor of v can be recolored with 2. Now we can let $f(v) = 1$ to obtain a coloring of H, which is a contradiction. □

Lemma 2.4. *Let $H = H(d_1, d_2)$ and let v be a 2-vertex of H where $N(v) = \{v_1, v_2\}$ and $d(v_1) \leq d_2 + 1$. If f is a coloring of $H - v$, then $f(v_1) = 1$ and $f(v_2) = 2$.*

Proof. If $f(v_1) = f(v_2)$, then letting $f(v) \in [2] \setminus \{f(v_1)\}$ gives a coloring of H, which is a contradiction. If $f(v_1) = 2$ and $f(v_2) = 1$, then let $f(v) = 2$ to obtain a coloring of H, unless v_1 is 2-saturated. This implies that $d(v_1) = d_2 + 1$ and $f(z) = 2$ for $z \in N(v_1) \setminus \{v\}$, so we can let $f(v_1) = 1$ to obtain a coloring of H, which is a contradiction. □