Discrete Mathematics 338 (2015) 43–46

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Blocking sets of Hermitian generalized quadrangles

Antonio Cossidente, Francesco Pavese *

Dipartimento di Matematica Informatica ed Economia, Università della Basilicata, Contrada Macchia Romana, I-85100 Potenza, Italy

ARTICLE INFO

Article history:
Received 24 January 2014
Received in revised form 24 June 2014
Accepted 23 August 2014
Available online 16 September 2014

Keywords:
Hermitian generalized quadrangle
Blocking set
Tight set
Cyclic spread
Unital

ABSTRACT

Some infinite families of minimal blocking sets on Hermitian generalized quadrangles are constructed.

1. Introduction

A finite classical polar space $\mathcal{P}$ of rank $r \geq 2$ arises from the set of all absolute points and totally isotropic (totally singular) subspaces of a polarity of a projective space $PG(n, q)$. The integer r denotes the vector dimension of a maximal totally isotropic (totally singular) subspace of $\mathcal{P}$, [2]. When $r = 2$ the polar space $\mathcal{P}$ is called a *classical generalized quadrangle* [11]. Examples of classical generalized quadrangles arise from the Hermitian surface $\mathcal{H}(3, q^2)$ of PG$(3, q^2)$ and the Hermitian variety $\mathcal{H}(4, q^2)$ of $PG(4, q^2)$. The generalized quadrangle $\mathcal{H}(3, q^2)$ has order (q^2, q): any line contains $q^2 + 1$ points and on any point there are $q + 1$ lines. The generalized quadrangle $\mathcal{H}(4, q^2)$ has order (q^2, q^3): any line contains $q^2 + 1$ points and on any point there are $q^3 + 1$ lines. An ovoid O of $\mathcal{P}$ is a point set of $\mathcal{P}$ which has exactly one point in common with every totally isotropic subspace of rank r.

A *blocking set* of $\mathcal{P}$ with respect to the set B of totally isotropic (totally singular) k-dimensional subspaces is a set of points of $\mathcal{P}$ that meets every k-dimensional subspace of B; a blocking set is *minimal* if the removal of any point leads to a non-blocking set (thus each point is *essential*). Of course, any ovoid of $\mathcal{P}$ is a trivial minimal blocking set of $\mathcal{P}$.

In this paper we are interested in blocking sets of the generalized quadrangles $\mathcal{H}(3, q^2)$ and $\mathcal{H}(4, q^2)$ with respect to generators, i.e., lines. We will construct several families of minimal blocking sets that are *non-linear* in the sense that they do not lie in a subspace of $PG(3, q^2)$ of dimension 2 and of $PG(4, q^2)$ of dimension 3, respectively.

There are plenty of examples of ovoids of $\mathcal{H}(3, q^2)$ [5]. The generalized quadrangle $\mathcal{H}(4, q^2)$ has no ovoids [13]. The size of an ovoid of $\mathcal{H}(3, q^2)$ ($\mathcal{H}(4, q^2)$) would be $q^3 + 1$ (resp. $q^5 + 1$) and this provides a lower bound for the size of a minimal blocking set. We are not aware of any upper bound for the size of a minimal blocking set besides the trivial one obtained by counting the number of generators.

Metsch conjectures that the smallest minimal blocking sets with respect to k-dimensional subspaces of Hermitian polar spaces $\mathcal{H}(n, q^2)$ are linear (i.e., they lie in a subspace of $\mathcal{H}(n, q^2)$ of dimension $n - k$). In [10] he proves the conjecture (with a minor caveat) when $k \leq (n - 3)/2$, and (with De Beule) in [8] he proves the conjecture (with a minor caveat) when n

* Corresponding author.
E-mail addresses: antonio.cossidente@unibas.it (A. Cossidente), francesco.pavese@unibas.it (F. Pavese).

http://dx.doi.org/10.1016/j.disc.2014.08.016

is even and $k < (n-1)/2$. For $k = (n-1)/2$ the conjecture remains unproven. Nevertheless linear blocking sets remain examples of small minimal blocking sets with sizes ranging from $(q^{(n+3)/2} - (-1)^{(n+3)/2})(q^{(n+1)/2} - (-1)^{(n+1)/2})/(q^2-1)$ to $q^{n-1}(q+1)$.

Blocking sets of Hermitian generalized quadrangles that are not ovoids appear rare. The following examples of minimal blocking sets of $\mathcal{H}(3, q^2)$ are known.

(a) Any ovoid.
(b) The set of points of $\mathcal{H}(3, q^2)$ in a tangent plane at an arbitrary point p of $\mathcal{H}(3, q^2)$, with the point p deleted, is a linear blocking set $\mathcal{B}$ of $\mathcal{H}(3, q^2)$ of size $q^3 + q^2$ [10].
(c) Let $\mathcal{U}$ be a Hermitian curve embedded in $\mathcal{H}(3, q^2)$ as a non-singular section with a plane π. Let $p \in \mathcal{U}$ and consider the q^2 non-tangent lines through p lying on π. Such lines, let us call them $L_1, \ldots, L_{q^2}$, are all secant to $\mathcal{H}(3, q^2)$. Consider r different lines L_i, $1 \le r \le q^2$ and let $H_i = L_i \cap \mathcal{H}(3, q^2)$. The set $\mathcal{B} := \mathcal{U} \setminus (\bigcup_{i=1}^{r} H_i) \bigcup_{i=1}^{r} H_i^{\perp} \cap \mathcal{H}(3, q^2)$, where $\perp$ is the polarity induced by the Hermitian surface, is a minimal blocking set of $\mathcal{H}(3, q^2)$ of size $q^3 + r$ [3].
(d) An infinite family for q odd, of size $q^3 + q^2$ admitting $\mathrm{PSL}(2, q)$ as an automorphism group [4].
(e) An infinite family for $q > 2$ even of size $(q+1)^3$, [6].

No other instances of small minimal blocking sets in $\mathcal{H}(3, q^2)$ are known apart from some computational examples and the examples in [3].

As far as we know no examples of non-linear minimal blocking sets of $\mathcal{H}(4, q^2)$ have been constructed so far.

2. Blocking sets of $\mathcal{H}(3, q^2)$

The symplectic group $\mathrm{PSp}(4, q)$ is embedded in $\mathrm{P\Gamma U}(4, q^2)$ as a subfield subgroup, stabilizing a subquadrangle of $\mathcal{H}(3, q^2)$ isomorphic to $\mathcal{W}(3, q)$. In terms of coordinates, assuming that $\mathcal{H}(3, q^2)$ has equation $X_1^{q+1} + X_2^{q+1} + X_3^{q+1} + X_4^{q+1} = 0$, where $X_1, \ldots, X_4$ are homogeneous projective coordinates in $\mathrm{PG}(3, q^2)$, the set $\{(x, \rho x^q, y, \rho y^q) : x, y \in \mathrm{GF}(q^2)\}$ with $\rho^{q+1} = -1$, is the point set of a symplectic Baer subgeometry $\mathcal{W}(3, q)$ embedded in $\mathcal{H}(3, q^2)$ [12]. Every generator of $\mathcal{H}(3, q^2)$ either meets $\mathcal{W}(3, q)$ in a totally isotropic line of $\mathcal{W}(3, q)$ or it is disjoint from it. This means that a point of $\mathcal{H}(3, q^2) \setminus \mathcal{W}(3, q)$ lies on a unique totally isotropic line of $\mathcal{W}(3, q)$. Under the Klein correspondence κ between lines of $\mathrm{PG}(3, q^2)$ and points of a Klein quadric $\mathcal{Q}^+(5, q^2)$ of $\mathrm{PG}(5, q^2)$, it turns out that $\mathcal{H}(3, q^2)$ corresponds to an elliptic quadric $\mathcal{Q}^-(5, q) \subset \mathcal{Q}^+(5, q^2)$ embedded in a Baer subgeometry $\mathrm{PG}(5, q)$ of $\mathrm{PG}(5, q^2)$ and that the symplectic subgeometry $\mathcal{W}(3, q)$ embedded in $\mathcal{H}(3, q^2)$ corresponds to a parabolic section $\mathcal{Q}(4, q)$ of $\mathcal{Q}^-(5, q)$. Thus the intersection of two symplectic subgeometries corresponds to the intersection of two such parabolic sections, which is a solid section. A solid meeting $\mathcal{Q}^-(5, q)$ in a hyperbolic quadric $\mathcal{Q}^+(3, q)$ contains $2(q+1)$ lines of the quadric. In this case, each regulus of the hyperbolic quadric corresponds to a hyperbolic line, and as the reguli are opposite, the two symplectic subgeometries intersects in a pair of hyperbolic lines, which are pairwise polar. It follows that the two symplectic subgeometries share $2(q+1)$ points forming a hyperbolic pair $\{\ell_1, \ell_2\}$. Also, there are exactly $q+1$ symplectic subgeometries embedded in $\mathcal{H}(3, q^2)$ sharing a hyperbolic pair and there exists a cyclic collineation group G of order $q+1$ in $P\Gamma U(4, q^2)$ permuting such symplectic subgeometries: in the dual setting G induces a Singer cycle of $\mathrm{PGL}(2, q)$ acting on $\mathcal{Q}^+(3, q)^{\perp}$, where $\perp$ is the non-degenerate orthogonal polarity induced by $\mathcal{Q}^-(5, q)$.

Theorem 2.1. *Let Σ_i, $i = 1, \ldots, q+1$, be the symplectic subgeometries embedded in $\mathcal{H}(3, q^2)$ sharing a hyperbolic pair ℓ_1, ℓ_2. Let $\mathcal{X}$ be a minimal blocking set of Σ_i for some $i = 1, \ldots, q+1$, of size k, such that $|\mathcal{X} \cap \ell_1| = k_1$ and $|\mathcal{X} \cap \ell_2| = k_2$. Assume that the generators of $\mathcal{H}(3, q^2)$ meeting both ℓ_1 and ℓ_2 are not tangent to $\mathcal{X}$ at points of $\ell_1 \cup \ell_2$. Then $(\mathcal{X} \setminus (\ell_1 \cup \ell_2))^G$ is a minimal blocking set of $\mathcal{H}(3, q^2)$ of size $(k - k_1 - k_2)(q+1)$.*

Proof. Let g be a generator of $\mathcal{H}(3, q^2)$ that does not meet both ℓ_1 and ℓ_2. Assume that g intersects Σ_i at $q+1$ points (a totally isotropic line, say r, of Σ_i). Then g is disjoint from Σ_j, $j \neq i$. Indeed, in the dual setting, r is a point, say P_r, of $\mathcal{Q}^-(5, q)$ on the parabolic quadric through $\mathcal{Q}^+(3, q)$ representing Σ_i. If r met Σ_j, $j \neq i$, at $q+1$ points, then P_r would also lie on the parabolic quadric representing Σ_j. It follows that P_r should lie on $\mathcal{Q}^+(3, q)$, a contradiction. Also, notice that the $q+1$ parabolic quadrics on $\mathcal{Q}^+(3, q)$ cover all points of $\mathcal{Q}^-(5, q)$. Let $\mathcal{X}$ be a minimal blocking set of Σ_i, for some $i = 1, \ldots, q+1$, of size k, such that $|\mathcal{X} \cap \ell_1| = k_1$ and $|\mathcal{X} \cap \ell_2| = k_2$. It follows that $(\mathcal{X} \setminus (\ell_1 \cup \ell_2))^G$ is a minimal blocking set of $\mathcal{H}(3, q^2)$ of the desired size. □

Remark 2.2. *Some examples*:

1. Let π be any plane of Σ_i. Then, either π intersects ℓ_1 and ℓ_2 in one point or π contains ℓ_i and meets ℓ_j in one point, $i \neq j$. In the first case we get a minimal blocking set of size $q^3 + 2q^2 - 1$. In the second case we get a minimal blocking set of size $q^3 + q^2 - q - 1$.
2. Assume that q is even and let $\mathcal{O}$ be any ovoid of Σ_i. Assume that ℓ_1 is secant to $\mathcal{O}$. We get a minimal blocking set of size $q^3 + q^2 - q - 1$.
3. Assume that q is even and let Q be a hyperbolic quadric of Σ_i such that the associated polarity is the symplectic polarity of Σ_i. We get either a minimal blocking set of size $q^3 + 3q^2 + 3q + 1$ (when $k_1 = k_2 = 0$) or a minimal blocking set of size $q^3 + 3q^2 - q - 3$ (when $k_1 = k_2 = 2$).