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Note

On testing Hamiltonicity of graphs

Alexander Barvinok

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1043, USA

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ABSTRACT

Let us fix a function $f(n) = o(n \ln n)$ and real numbers $0 \leq \alpha < \beta \leq 1$. We present a polynomial time algorithm which, given a directed graph G with n vertices, decides either that one can add at most βn new edges to G so that G acquires a Hamiltonian circuit or that one cannot add αn or fewer new edges to G so that G acquires at least $e^{-f(n)}n!$ Hamiltonian circuits, or both.

1. Introduction and main results

Let $G = (V, E)$ be a directed graph with set V of vertices and set E of edges. A *Hamiltonian circuit* is a closed walk $i_1 \to i_2 \to \cdots \to i_n \to i_1$ that visits every vertex of G exactly once. It is a classical NP-complete problem to determine whether a given directed graph contains a Hamiltonian circuit (in which case the graph is called *Hamiltonian*). In what follows, n denotes the number of vertices of the graph, $n = |V|$.

The following version of the problem is also known to be NP-complete: Given $0 < \beta < 1$, is it true that one can add at most βn new edges to a given directed graph with n vertices so that the graph becomes Hamiltonian? In fact, for any fixed $\beta < 1/320$, the problem is NP-complete; see [9,4].

Anastasios Sidiropoulos pointed out to the author that testing Hamiltonicity does not become any easier if we are promised that should the directed graph be Hamiltonian, it contains at least $\exp\{-n^\epsilon\}\, n!$ Hamiltonian circuits for some fixed $\epsilon > 0$. Indeed, let G be a given directed graph with m vertices. Let us choose $k > 2/\epsilon$ and construct a new directed graph $\widehat{G}$ by attaching a complete directed graph with m^k vertices by two edges to two selected vertices u and v of G. Hence the new graph $\widehat{G}$ with $n = m + m^k$ vertices contains at least $(m^k - 2)!$ Hamiltonian circuits if and only if G contains a Hamiltonian path with endpoints u and v. If there is no such path in G then $\widehat{G}$ contains no Hamiltonian circuits.

Let us choose a function $f(n) = o(n \ln n)$ and fix two numbers $0 \leq \alpha < \beta \leq 1$. We present a polynomial time algorithm, which, given a directed graph G with n vertices, outputs at least one of the following two statements (a) and (b):

(a) one can add at most βn new edges to G so that G acquires a Hamiltonian circuit;
(b) one cannot add αn or fewer new edges to G so that G acquires at least $e^{-f(n)}n!$ Hamiltonian circuits.

For example, confronted with two directed graphs on n vertices one of which contains at least $10^{-3n}n!$ Hamiltonian circuits and the other does not become Hamiltonian unless more than $10^{-3}n$ new edges added to the graph, our algorithm will be able to tell which graph is which in polynomial time. On the other hand, testing whether one needs to add at least $10^{-3}n$ new edges to a given directed graph on n vertices so that the graph becomes Hamiltonian is an NP-hard problem and testing whether a given directed graph on n vertices contains at least $10^{-3n}n!$ Hamiltonian circuits is also an NP-hard problem.

E-mail address: barvinok@umich.edu.

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It may happen though that while the statements (a) and (b) above are both true, the algorithm outputs only one of them.

We note that even if we are told that the graph contains at least $e^{-f(n)}n!$ Hamiltonian circuits, it is not obvious how to construct any of the circuits efficiently (deterministically or probabilistically). We also note some vague similarity with property testing questions [5].

Our algorithm is based on computing permanents and their Hamiltonian versions.

1.1. Permanents and Hamiltonian permanents

Let $A = \left(a_{ij}\right)$ be an $n \times n$ real matrix. The *permanent of* A is defined as

$$\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the sum is taken over the symmetric group S_n of permutations of the set $\{1, \ldots, n\}$. As is known, the problem of computing the permanent exactly is #*P*-hard, even if the entries of A are restricted to be 0 and 1 [11]. For non-negative matrices a fully polynomial randomized approximation scheme is available [6]. We, however, are interested in computing permanents of a rather restricted class of matrices. Namely, let us suppose that

$$\frac{1}{n^{0.1}} \leq a_{ij} \leq 1 \quad \text{for all } i, j. \tag{1}$$

Then a version of the scaling algorithm of [8], see also [3], approximates $\operatorname{per} A$ in polynomial in n time within an $O\left(\exp\left\{n^{0.35}\right\}\right)$ factor. The algorithm is deterministic and easy to implement. We review the algorithm in Section 3.

Let $H_n \subset S_n$ be the subset of $(n-1)!$ permutations consisting of a single cycle. We define the *Hamiltonian permanent* by

$$\operatorname{ham} A = \sum_{\sigma \in H_n} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

If A is a 0–1 matrix then it is an NP-complete problem to tell $\operatorname{ham} A$ from 0, as the problem is equivalent to testing Hamiltonicity of the directed graph with adjacency matrix A. It turns out, however, that when (1) holds, $\operatorname{per} A$ and $\operatorname{ham} A$ have the same logarithmic order.

Theorem 1. *Let $A = \left(a_{ij}\right)$ be an $n \times n$ matrix such that*

$$\epsilon \leq a_{ij} \leq 1$$

for some $\epsilon > 0$ and all i, j. Let

$$r = \left\lfloor \frac{4 \ln n}{\epsilon^2} \right\rfloor + 6.$$

Then

$$\frac{1}{2r} \left(\frac{\epsilon}{n}\right)^r \operatorname{per} A \leq \operatorname{ham} A \leq \operatorname{per} A.$$

In particular, if we choose $\epsilon = n^{-0.1}$ then $\operatorname{per} A$ approximates $\operatorname{ham} A$ within an $O\left(\exp\left\{n^{0.3}\right\}\right)$ factor.

We prove Theorem 1 in Section 2.

In a different setting, the relation between the permanent and Hamiltonian permanent of the adjacency matrix of a k-regular graph was used in [12] while the first use of permanents to bound the number of Hamiltonian circuits in tournaments goes back to [1].

1.2. Testing Hamiltonicity of graphs

Let us fix a function $f(n) = o(n \ln n)$ and real numbers $0 \leq \alpha < \beta \leq 1$. Given a directed graph $G = (V, E)$, we identify $V = \{1, \ldots, n\}$ and construct an $n \times n$ matrix $A = A(G), A = \left(a_{ij}\right)$, as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } (i \to j) \in E \\ n^{-0.1} & \text{otherwise.} \end{cases}$$

Using Theorem 1 and the algorithm of Section 3, we compute $\operatorname{ham} A$ within a factor of $O\left(\exp\left\{n^{0.4}\right\}\right)$. If G does not become Hamiltonian unless more than βn new edges are added to G, then

$$\operatorname{ham} A \leq n^{-0.1\beta n} n! \tag{2}$$

If, however, one can add αn or fewer edges to G so that G acquires at least $e^{-f(n)}n!$ Hamiltonian circuits then

$$\operatorname{ham} A \geq n^{-0.1\alpha n} e^{-f(n)} n! \tag{3}$$