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Polyhedral study of the connected subgraph problem

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ABSTRACT

In this paper, we study the connected subgraph polytope which is the convex hull of the solutions to a related combinatorial optimization problem called the maximum weight connected subgraph problem. We strengthen a cut-based formulation by considering some new partition inequalities for which we give necessary and sufficient conditions to be facet defining. Based on the separation problem associated with these inequalities, we give a complete polyhedral characterization of the connected subgraph polytope on cycles and trees.

1. Introduction

Given an undirected graph G with vertex set V and edge set E, a subgraph H of G is *connected* if for any two vertices of H, there exists a path connecting them [18]. This paper deals with the connected subgraph polytope, that is, the convex hull of the incidence vectors of edge sets inducing connected subgraphs of G. Given an undirected graph G and a real-valued edge-weight vector, the *Maximum Weight Connected Subgraph Problem*, hereafter denoted MWCSP, consists of finding a maximum-weight subset of edges which induces a connected subgraph of G.

MWCSP was first considered by Kerivin and Ng [15] who focused on its complexity. They showed that it is a NP-hard combinatorial optimization problem even on planar or bipartite graphs by reducing the Steiner Tree problem to MWCSP. Kerivin and Ng [15] also proved that the MWCSP and the prize-collecting Steiner tree problem as defined in Johnson et al. [14] and in Goemans and Williamson [10], are equivalent optimization problems. Using a similar approach as Feigenbaum et al. [8], they showed that it is NP-hard to approximate MWCSP to within a constant factor. What makes MWCSP look like a non-trivial optimization problem on classes of graphs where the Steiner Tree problem is trivial is the fact that MWCSP deals with both positive and negative weights on the edges.

The main reasons for studying the connected subgraph polytope are that Kerivin and Ng [15] showed that its related optimization problem, MWCSP, is solvable in polynomial time on certain classes of graphs such as trees, cycles, and fans. Consequently, it is of interest to know if there are ways to describe or generate facet-inducing inequalities to the polyhedron that describes all the feasible solutions to MWCSP on these classes of graphs.

This article is organized as follows. In Section 2, we present the Connected Subgraph Polytope, denoted CSP(G), which corresponds to the convex hull of the incidence vectors of connected subgraphs of G. We then give a formulation for CSP(G),

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and necessary and sufficient conditions for the facet-defining inequalities. We conclude Section 2 by giving a complete polyhedral characterization of CSP(G) on graphs having no matching of cardinality three. Section 3 introduces new valid inequalities, called the *matching-partition inequalities*, and gives necessary and sufficient conditions for these inequalities to be facet defining. In Section 4, we investigate the separation problems for all the discussed families of inequalities. In Section 5, we give a complete polyhedral characterization of CSP(G) on cycles and trees. Finally, some concluding remarks are given in Section 6.

We conclude this introduction with some definitions and notation, which have been mainly taken from [7,18].

Let G be a simple, connected, and undirected graph with vertex set $V(G)$ and edge set $E(G)$; when there is no confusion on which graph we are describing, we will label the graph as $G = (V, E)$. The *order* n of G is its number of vertices, that is, $n = |V|$. The number of edges of G is denoted by m. If $e \in E$ is an edge with extremities u and v, we also write uv to denote e.

A path, cycle, and complete graph of order n are denoted P_n, C_n, and K_n, respectively. If $n \geq 3$, a *star* or *claw* is the complete bipartite graph $K_{1,n}$, that is, the graph having one vertex being adjacent to the other n vertices. The *line graph* $L(G)$ of G is the graph on E wherein $e, f \in E$ are adjacent as vertices in $L(G)$ if and only if they are adjacent as edges in G.

Let U be a subset of V. The set of edges having one extremity in U and the other one in $\overline{U} = V \setminus U$ is called a *cut* and is denoted by $\delta(U)$. If $U = \{v\}$ for some $v \in V$, then we write $\delta(v)$ for $\delta(\{v\})$. We denote by $E[U]$ the set of edges having both extremities in U and $G[U]$ the subgraph induced by U (*i.e.*, $G[U] = (U, E[U])$). Given $W \subset V$ with $W \cap U = \emptyset$, $[U, W]$ denotes the set of edges having one extremity in U and the other one in W. If $\pi = \{V_1, \ldots, V_p\}$, $p \geq 2$, is a partition of V, then we denote by $E(\pi)$ the set of edges having their extremities in different classes of π. We may also write $\delta(V_1, \ldots, V_p)$ for $E(\pi)$.

Let $F \subseteq E$. Given $\mathbf{x} \in \mathbb{R}^E$, $\mathbf{x}(F)$ will denote $\sum_{e \in F} x(e)$.

2. The connected subgraph polytope and preliminaries

Given any edge set $F \subseteq E$, its *incidence vector* is the vector $\mathbf{x}^F$ in $\{0, 1\}^E$ such that $x^F_e = 1$ if and only if $e \in F$. The *connected subgraph polytope* is the convex hull of the incidence vectors of edge sets inducing connected subgraphs of G, that is,

$$\mathrm{CSP}(G) = \mathrm{conv}\{\mathbf{x}^F \in \{0, 1\}^E : G[F] \text{ is connected}\}.$$

We first get rid of the trivial cases associated with $n = 2$ (i.e., $G = P_2$). In fact, $\mathrm{CSP}(P_2) = P(P_2) = [0, 1]$. Consequently, henceforth and unless otherwise mentioned, we suppose that G is of order at least 3.

The incidence vector of a connected subgraph of G must satisfy the following *connectivity inequalities*

$$\mathbf{x}(\delta(W)) \geq x_l + x_r - 1 \quad \text{for all } (W, l, r) \in \mathcal{C}(G), \tag{1}$$

where

$$\mathcal{C}(G) = \{(W, l, r) : \emptyset \neq W \subset V, l \in E[W], \text{ and } r \in E[\overline{W}]\}.$$

For $F \subseteq E$, if $\mathbf{x}^F$ satisfies inequalities (1), then there exists at least a path in $G[F]$ connecting any pair of non-adjacent edges l and r in F. Let $P(G)$ be the set of vectors in $\mathbb{R}^E$ which satisfy the connectivity inequalities (1) and the *box inequalities*

$$0 \leq x_e \leq 1 \quad \text{for all } e \in E. \tag{2}$$

The next proposition immediately follows, and then is given with no proof.

Proposition 1. *Polytope $P(G) = \{\mathbf{x} \in \mathbb{R}^E : \mathbf{x}$ satisfies (1)–(2)$\}$ is a formulation for $CSP(G)$, that is, $CSP(G) = P(G) \cap \mathbb{Z}^E$.*

Before studying the polyhedral structure of this polytope, we address the situation where the all-zero vector is excluded from CSP(G). In fact, according to Diestel [7], a connected subgraph cannot be the empty graph. To encompass this restriction, we consider the polytope

$$\mathrm{CSP}'(G) = \mathrm{conv}\{\mathbf{x}^F \in \{0, 1\}^E : G[F] \text{ is connected and } |F| \geq 1\}$$

and the *non-emptyness inequality*

$$\mathbf{x}(E) \geq 1. \tag{3}$$

Note that $\mathrm{CSP}'(G) \subseteq \mathrm{CSP}(G)$. As stated in the next proposition, CSP(G) and CSP$'(G)$ are full dimensional, and necessary and sufficient conditions for inequalities (2) to be facet-defining of both polytopes are established. The proofs of these results are omitted, since they use standard techniques. (See [6] for detailed proofs.)

Proposition 2. *Let $G = (V, E)$ be a simple, connected, and undirected graph of order at least* 3.

(i) *The polytopes $CSP(G)$ and $CSP'(G)$ are full dimensional.*
(ii) *For any edge $e \in E$, the inequality $x_e \leq 1$ defines a facet of $CSP(G)$ and $CSP'(G)$.*
(iii) *For any edge $e \in E$, the inequality $x_e \geq 0$ defines a facet of $CSP(G)$.*