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Characterizing paths graphs on bounded degree trees by minimal forbidden induced subgraphs

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ABSTRACT

An undirected graph G is called a VPT graph if it is the vertex intersection graph of a family of paths in a tree. The class of graphs which admit a VPT representation in a host tree with maximum degree at most h is denoted by $[h, 2, 1]$. The classes $[h, 2, 1]$ are closed under taking induced subgraphs, therefore each one can be characterized by a family of minimal forbidden induced subgraphs. In this paper we associate the minimal forbidden induced subgraphs for $[h, 2, 1]$ which are VPT with (color) h-critical graphs. We describe how to obtain minimal forbidden induced subgraphs from critical graphs, even more, we show that the family of graphs obtained using our procedure is exactly the family of VPT minimal forbidden induced subgraphs for $[h, 2, 1]$. The members of this family together with the minimal forbidden induced subgraphs for VPT (Lévêque et al., 2009; Tondato, 2009), are the minimal forbidden induced subgraphs for $[h, 2, 1]$, with $h \geq 3$. By taking $h = 3$ we obtain a characterization by minimal forbidden induced subgraphs of the class VPT $\cap$ EPT $=$ EPT $\cap$ Chordal $= [3, 2, 2] = [3, 2, 1]$ (see Golumbic and Jamison, 1985).

1. Introduction

The **intersection graph** of a family is a graph whose vertices are the members of the family, and the adjacency between vertices is defined by a non-empty intersection of the corresponding sets. Classic examples are interval graphs and chordal graphs.

An **interval graph** is the intersection graph of a family of intervals of the real line, or, equivalently, the vertex intersection graph of a family of subpaths of a path. A **chordal graph** is a graph without chordless cycles of length at least four. Gavril [5] proved that a graph is chordal if and only if it is the vertex intersection graph of a family of subtrees of a tree. Both classes have been widely studied [3].

In order to allow larger families of graphs to be represented by subtrees, several graph classes are defined imposing conditions on trees, subtrees and intersection sizes [9,10]. Let h, s and t be positive integers; an (h, s, t)-representation of a graph G consists in a host tree T and a collection $(T_v)_{v \in V(G)}$ of subtrees of T, such that (i) the maximum degree of T is at most h, (ii) every subtree T_v has maximum degree at most s, and (iii) two vertices v and v' are adjacent in G if and only if the corresponding subtrees T_v and $T_{v'}$ have at least t vertices in common in T. The class of graphs that have an (h, s, t)-representation is denoted by $[h, s, t]$. When there is no restriction on the maximum degree of T or on the maximum degree of the subtrees, we use $h = \infty$ and $s = \infty$ respectively. Therefore, $[\infty, \infty, 1]$ is the class of chordal graphs and $[2, 2, 1]$ is the class of interval graphs. The classes $[\infty, 2, 1]$ and $[\infty, 2, 2]$ are called VPT and EPT respectively in [7]; and UV and UE, respectively in [13].

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In [6,14], it is shown that the problem of recognizing VPT graphs is polynomial time solvable. Recently, in [1], generalizing a result given in [7], we have proved that the problem of deciding whether a given VPT graph belongs to $[h, 2, 1]$ is NP-complete even when restricted to the class VPT $\cap$ Split without dominated stable vertices. The classes $[h, 2, 1]$, $h \geq 2$, are closed under taking induced subgraphs; therefore each one can be characterized by a family of minimal forbidden induced subgraphs. Such a family is known only for $h = 2$ [11] and there are some partial results for $h = 3$ [4]. In this paper we associate the VPT minimal forbidden induced subgraphs for $[h, 2, 1]$ with (color) h-critical graphs. We describe how to obtain minimal forbidden induced subgraphs from critical graphs, even more, we show that the family of graphs obtained using our procedure is exactly the family of VPT minimal forbidden induced subgraphs for $[h, 2, 1]$. The members of this family together with the minimal forbidden induced subgraphs for VPT (see Fig. 2) which were determined in [12,15], are the minimal forbidden induced subgraphs for $[h, 2, 1]$, with $h \geq 3$. Notice that by taking $h = 3$ we obtain a characterization by minimal forbidden induced subgraphs of the class VPT $\cap$ EPT $=$ EPT $\cap$ Chordal $= [3, 2, 2] = [3, 2, 1]$ [7].

The paper is organized as follows: in Section 2, we provide basic definitions and basic results. In Section 3, we give necessary conditions for VPT minimal non-$[h, 2, 1]$ graphs. In Section 4, we show a procedure to construct minimal non-$[h, 2, 1]$ graphs. In Section 5, we describe the family of all minimal non-$[h, 2, 1]$ graphs.

2. Preliminaries

Throughout this paper, graphs are connected, finite and simple. The **vertex set** and the **edge set** of a graph G are denoted by $V(G)$ and $E(G)$ respectively. The **open neighborhood** of a vertex v, represented by $N_G(v)$, is the set of vertices adjacent to v. The **closed neighborhood** $N_G[v]$ is $N_G(v) \cup \{v\}$. The **degree** of v, denoted by $d_G(v)$, is the cardinality of $N_G(v)$. For simplicity, when no confusion can arise, we omit the subindex G and write $N(v)$, $N[v]$ or $d(v)$. Two vertices $x, y \in V(G)$ are called **true twins** if $N[x] = N[y]$.

A **complete set** is a subset of mutually adjacent vertices. A **clique** is a maximal complete set. The family of cliques of G is denoted by $\mathcal{C}(G)$. A **stable set**, also called an independent set, is a subset of pairwise non-adjacent vertices.

A graph G is **k-colorable** if its vertices can be colored with at most k colors in such a way that no two adjacent vertices share the same color. The **chromatic number** of G, denoted by $\chi(G)$, is the smallest k such that G is k-colorable. A vertex $v \in V(G)$ or an edge $e \in E(G)$ is a **critical element** of G if $\chi(G - v) < \chi(G)$ or $\chi(G - e) < \chi(G)$ respectively. A graph G with chromatic number h is **h-vertex critical** (resp. **h-critical**) if each of its vertices (resp. edges) is a critical element.

A **VPT representation** of G is a pair $\langle \mathcal{P}, T\rangle$ where $\mathcal{P}$ is a family $(P_v)_{v \in V(G)}$ of subpaths of a host tree T satisfying that two vertices v and v' of G are adjacent if and only if P_v and $P_{v'}$ have at least one vertex in common; in such case we say that P_v intersects $P_{v'}$. When the maximum degree of the host tree is at most h the VPT representation of G is called an $(h, 2, 1)$-representation of G. The class of graphs which admit an $(h, 2, 1)$-representation is denoted by **[h, 2, 1]**.

Since a family of vertex paths in a tree satisfies the Helly property [2], if C is a clique of G then there exists a vertex q of T such that $C = \{v \in V(G) \mid q \in V(P_v)\}$. On the other hand, if q is any vertex of the host tree T, the set $\{v \in V(G) \mid q \in V(P_v)\}$, denoted by $\mathbf{C_q}$, is a complete set of G, but not necessarily a clique. In order to avoid this drawback, we introduce the notion of full representation at q.

Let $\langle \mathcal{P}, T\rangle$ be a VPT representation of G and let q be a vertex of T with degree h. The connected components of $T - q$ are called the **branches of T at q**. A path is **contained** in a branch if all its vertices are vertices of the branch. Notice that if $N_T(q) = \{q_1, q_2, \ldots, q_h\}$, then T has exactly h branches at q. The branch containing q_i is denoted by $\mathbf{T_i}$. Two branches T_i and T_j are **linked** by a path $P_v \in \mathcal{P}$ if both vertices q_i and q_j belong to $V(P_v)$.

Definition 1. A VPT representation $\langle \mathcal{P}, T\rangle$ is **full at a vertex q** of T if, for every two branches T_i and T_j of T at q, there exist paths $P_v, P_w, P_u \in \mathcal{P}$ such that: (i) the branches T_i and T_j are linked by P_v; (ii) P_w is contained in T_i and intersects P_v in at least one vertex; and (iii) P_u is contained in T_j and intersects P_v in at least one vertex.

A clear consequence of the previous definition is that if $\langle \mathcal{P}, T\rangle$ is full at a vertex q of T, with $d_T(q) = h \geq 3$, then C_q is a clique of G.

The following theorem from [1] shows that any VPT representation which is not full at some vertex q of T with $d_T(q) \geq 4$ can be modified to obtain a new VPT representation without increasing the maximum degree of the host tree; while decreasing the degree of the vertex q.

Theorem 2 *([1]). Let $\langle \mathcal{P}, T\rangle$ be a VPT representation of G. Assume there exists a vertex $q \in V(T)$ with $d_T(q) = h \geq 4$ and two branches of T at q which are linked by no path of $\mathcal{P}$. Then there exists a VPT representation $\langle \mathcal{P}', T'\rangle$ of G with $V(T') = V(T) \cup \{q'\}$, $q' \notin V(T)$, and*

$$d_{T'}(x) = \begin{cases} 3 & \text{if } x = q'; \\ h - 1 & \text{if } x = q; \\ d_T(x) & \text{if } x \in V(T') \setminus \{q, q'\}. \end{cases}$$