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On large semi-linked graphs

Alexander Halperin [a,*], Colton Magnant [b], Hua Wang [b]

[a] Department of Mathematics and Computer Science, Salisbury University, 1101 Camden Ave., Salisbury, MD 21804, United States

[b] Department of Mathematical Sciences, Georgia Southern University, 65 Georgia Ave., Room 3008, P.O. Box 8093, Statesboro, GA 30460, United States

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ABSTRACT

Let H be a multigraph, possibly with loops, and consider a set $S \subseteq V(H)$. A (simple) graph G is (H, S)-*semi-linked* if, for every injective map $f : S \to V(G)$, there exists an injective map $g : V(H) \setminus S \to V(G) \setminus f(S)$ and a set of $|E(H)|$ internally disjoint paths in G connecting pairs of vertices of $f(S) \cup g(V(H) \setminus S)$ for every edge between the corresponding vertices of H. This new concept of (H, S)-semi-linkedness is a generalization of H-*linkedness*. We establish a sharp minimum degree condition for a sufficiently large graph G to be (H, S)-semi-linked.

1. Introduction

For all basic definitions and notation, see [2]. Let H be a multigraph, possibly with loops unless stated otherwise. We use the notation $f : A \hookrightarrow B$ to denote an injective map f. A graph G is said to be H-*linked* if for every mapping $f : V(H) \hookrightarrow V(G)$, there is a mapping of the edges of H to paths of G that are vertex-disjoint except for the ends, which are the vertices $w \in G$ such that $f(v) = w$ for some $v \in H$. Such a mapping [14] is called an H-*subdivision* (or *topological minor* [3]) in G. For brevity, let $|H| = |V(H)|$ and $e(H) = |E(H)|$.

The idea of H-linkedness was first introduced by Jung in [9] and then developed in [8,15]. Since then, there have been recent developments on the minimum degree criteria for a graph to be H-linked. Let H be a connected *loopless* multigraph. Kostochka and Yu determined in [11] that a graph G of order $n \geq 5e(H) + 6$ with $\delta(G) \geq \frac{n+e(H)-2}{2}$ is H-linked, and that the lower bound on $\delta(G)$ is sharp for bipartite H. Let $B(H)$ be the number of edges in a maximum edge-cut of H. The same authors later proved in [12] that if $\delta(H) \geq 2$, then a graph G of order $n \geq 7.5e(H)$ with $\delta(G) \geq \frac{n+B(H)-2}{2}$ is H-linked, and that the lower bound on $\delta(G)$ is sharp. Now assume (for the rest of the paper) that H *may contain* loops. Ferrara, Gould, Tansey, and Whalen showed in [5] that a sufficiently large graph G of order n satisfying the sharp condition $\delta(G) \geq \frac{n+B(H)-2}{2}$ is H-linked. Note that the inclusion of loops in H comes at the expense of a linear (or any reasonable) lower bound on n, although this is not to say such a lower bound is best-possible. The results from [12,5] were united in [7], and the sharp condition on $\delta(G)$ for G to be H-linked was generalized to include *disconnected* multigraphs H. Most importantly for our purposes, the term $b(H)$ was defined in [4] by Ferrara et al. as

$$b(H) = \max_{\substack{R \cup N \cup U = V(H) \\ e(R,U) \geq 1}} \left\{ |N| + e(R, U) \right\}. \quad (1)$$

In [4], Ferrara et al. generalized the main result of [7] by proving a sharp condition involving $\sigma_2(G)$, the minimum degree sum (of nonadjacent vertices) of G, for G to be H-linked. Let h_0 denote the number of isolated vertices in H. Ferrara et al. showed

* Corresponding author.
E-mail addresses: adhalperin@salisbury.edu (A. Halperin), cmagnant@georgiasouthern.edu (C. Magnant), hwang@georgiasouthern.edu (H. Wang).

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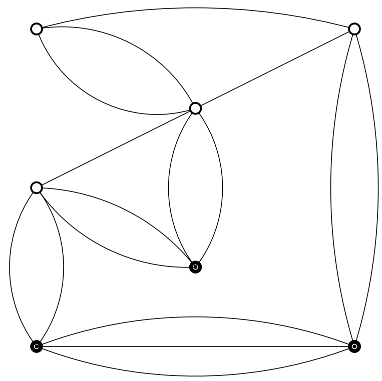

H

Fig. 1. A multigraph H with S consisting of the three solid vertices.

that a graph G of order n with $\delta(G) \geq 4e(H) + h_0$ and $\sigma_2(G) \geq n + b(H) - 2$ is H-linked. Although the sharp examples for each of these minimum degree conditions all hinge on the connectivity of G, sharp connectivity criteria for H-linkedness remain unknown.

A special case of H-linkedness that is more directly related to connectivity is the concept of k-linkedness. A graph G is *k-linked* if for every choice of $2k$ distinct vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$, there exists a set of disjoint (s_i, t_i)-paths for all i. It was shown in [13] that a graph G is k-linked if either G is $2k$-connected and has at least $5k|G|$ edges or if G is $10k$-connected, the latter being our Theorem 2. The former result improved Theorem 1.3 in [10], which states that a $2k$-connected graph G is k-linked if G has average degree at least $12k$. The main result in [10], however, consisted of minimum degree and degree-sum bounds for a graph on n vertices to be k-linked. Both [13,10] were published around the same time and borrowed ideas from each other, hence the similarity of their results and methodology.

We generalize the concept of H-linkedness to include subdivisions where only a certain set of vertices in $V(H)$ is mapped into $V(G)$ arbitrarily. Let G be a graph and let $\mathscr{P}(G)$ be the set of paths in G. Suppose we are given a multigraph H and a subset $S \subseteq V(H)$. Whenever we define $S \subseteq V(H)$, let $T = V(H) \setminus S$. A graph G is *(H, S)-semi-linked* if, for every map $f : S \hookrightarrow V(G)$, there exists a map $g : T \hookrightarrow V(G) \setminus f(S)$ and a set of $|E(H)|$ internally disjoint paths $\mathscr{P}(f, g) \subseteq \mathscr{P}(G)$ connecting vertices of $f(S) \cup g(T)$ for every edge between the corresponding vertices of H. Given f and g, such an H-subdivision of G containing $\mathscr{P}(f, g)$ is called an *(H, S)-semi-linkage*. (Note that $\mathscr{P}(f, g)$ contains $f(S) \cup g(T)$ as well.) Call $f(S) \cup g(T)$ the set of *ground vertices* and the paths in $\mathscr{P}$ *edge-paths*. Other authors (see [5,6]) have used the same terminology for an H-subdivision in an H-linked graph. In this paper, when we refer to ground vertices and edge-paths, we are referring only to those of an (H, S)-semi-linkage.

It should be noted that (H, S)-semi-linkedness completes the spectrum of linkedness between a graph and an H-subdivision. On the one hand, an $(H, \emptyset)$-semi-linked graph contains an H-subdivision (and perhaps no others), while on the other hand, an $(H, V(H))$-semi-linked graph is H-linked. In general, if a graph G is (H, S)-semi-linked, then we can always guarantee the existence of an H-subdivision in G on $|S|$ arbitrary vertices in $V(G)$.

We now define $s(H, S)$, a generalization of $b(H)$ that is used in our main result. Suppose we are given a multigraph H and a subset $S \subseteq V(H)$. Let c_S and c_T be (possibly improper) colorings of S and T, respectively, using the color set {red, blue, green}. Let R, U, and N be the sets of red, blue, and green vertices in $V(H)$, respectively, and let $e(A, B)$ denote the number of edges between (disjoint) vertex sets A and B. Finally, recall that $\kappa(G)$ denotes the connectivity of G.

It is a known result [1] that a graph G with minimum degree $\delta(G)$ and connectivity $\kappa(G)$ satisfies $\kappa(G) \geq 2\delta(G) + 2 - n$. I.e., if $\delta(G) \geq \frac{n+a-2}{2}$ for $a \geq 0$, then $\kappa(G) \geq a$. Define

$$s(H, S) = \max_{c_S} \min_{c_T} \left\{|N| + e(R, U)\right\} - 2;$$

the condition $\delta(G) \geq \frac{n+s(H,S)}{2}$ then guarantees $\kappa(G) \geq s(H, S) + 2 = \max_{c_S} \min_{c_T} \left\{|N| + e(R, U)\right\}$.

The following example gives a calculation of $s(H, S)$ for a multigraph H and vertex set $S \subseteq V(H)$.

Example 1. Consider the multigraph H and vertex set $S \subseteq V(H)$ in Fig. 1. Considering all colorings c_S and all subsequent colorings c_T, we find that $c'_S =$ {red, green, blue} and $c'_T =$ {blue, red, blue, blue}, together with their green, red, and blue sets N', R', and U', respectively, produce the value $|N'| + e(R', U') = s(H, S) + 2 = \max_{c_S} \min_{c_T} \left\{|N| + e(R, U)\right\} = 1 + 4 = 5$ (see Fig. 2).

Any graph G with a minimum cutset C satisfying $|C| < 5$ is not (H, S)-semi-linked, as the green vertex in H may be mapped into C and the 4 red-to-blue edges in H may be mapped to internally disjoint paths in G, each passing through C. □