Discrete Mathematics 326 (2014) 29–33

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Long geodesics in subgraphs of the cube

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ARTICLE INFO

Article history:
Received 23 January 2013
Received in revised form 13 February 2014
Accepted 14 February 2014
Available online 18 March 2014

Keywords:
Geodesic
Hypercube
Antipodal colouring

ABSTRACT

We show that any subgraph of the hypercube Q_n of average degree d contains a geodesic of length d, where by geodesic we mean a shortest path in Q_n. This result, which is best possible, strengthens a theorem of Feder and Subi. It is also related to the 'antipodal colourings' conjecture of Norine.

1. Introduction

Given a graph G of average degree d, a classic result of Dirac [3] guarantees that G contains a path of length d. Moreover, for general graphs this is the best possible bound, as can be seen by taking G to be K_{d+1}, the complete graph on $d+1$ vertices.

The hypercube Q_n has vertex set $\{0, 1\}^n$ and two vertices $x, y \in Q_n$ are joined by an edge if they differ on a single coordinate. In [9] a similar question was considered for subgraphs of the hypercube Q_n. That is, given a subgraph G of Q_n of average degree d, how long a path must G contain? The main result was the following theorem.

Theorem 1.1 *([9]). Every subgraph G of Q_n of minimum degree d contains a path of length $2^d - 1$.*

Combining Theorem 1.1 with the standard fact that any graph of average degree d contains a subgraph with minimum degree at least $d/2$, we see that any subgraph G of Q_n with average degree d contains a path of length at least $2^{d/2} - 1$.

In this paper we consider the analogous question for geodesics. A path in Q_n is a geodesic if it forms a shortest path in Q_n between its endpoints. Equivalently, a path is a geodesic if no two of its edges have the same direction, where an edge $xy \in E(Q_n)$ is said to have direction i when x and y differ in coordinate i. Given a subgraph G of Q_n of average degree d, how long a geodesic must G contain?

It is trivial to see that any such graph must contain a geodesic of length $d/2$. Indeed, taking a subgraph G' of G with minimal degree at least $d/2$ and starting from any vertex of G', we can greedily pick a geodesic of length $d/2$ by choosing a new edge direction at each step.

On the other hand the d-dimensional cube Q_d shows that, in general, we cannot find a geodesic of length greater than d in G. Our main result is that this upper bound is sharp.

Theorem 1.2. *Every subgraph G of Q_n of average degree d contains a geodesic of length at least d.*

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http://dx.doi.org/10.1016/j.disc.2014.02.013
0012-365X/

Since the endpoints of the geodesic in G guaranteed by Theorem 1.2 are at Hamming distance at least d, Theorem 1.2 extends the following result of Feder and Subi [4].

Theorem 1.3 ([4]). *Every subgraph G of Q_n of average degree d contains two vertices at Hamming distance d apart.*

We remark that neither Theorem 1.2 nor Theorem 1.3 follow from isoperimetric considerations alone. Indeed, if G is a subgraph of Q_n of average degree d, by the edge isoperimetric inequality for the cube ([1,5,6,8]; see [2] for background) we have $|G| \geq 2^d$. However if n is large, a Hamming ball of small radius may have size larger than 2^d without containing a long geodesic.

While Theorem 1.2 implies Theorem 1.3, we have also given an alternate proof of Theorem 1.3 from a result of Katona [7] which we feel may be of interest. The proofs of Theorems 1.2 and 1.3 are given in Sections 2 and 3 respectively.

Finally, Feder and Subi's theorem was motivated by a conjecture of Norine [10] on antipodal colourings of the cube. In the last section of this short paper we discuss Theorem 1.2 in relation to Norine's conjecture.

Notation: our notation is standard. Given a graph G, let $|G|$ denote the number of vertices of G and let $E(G)$ denote the edge set of G. Given a path $P = x_0 \dots x_l$, we say that P has length l and denote this by writing $|P| = l$. Given a path $P = x \cdots y$ and a vertex $z \notin V(P)$, we write Pyz to denote the path obtained by adjoining the edge yz to P. Given a set X, we write $\mathcal{P}(X)$ for its power set and $X^{(k)}$ for the set of subsets of X of size k. For $n \in \mathbb{N}$, let $[n] = \{1, \dots, n\}$.

2. Proofs of Theorems 1.2 and 1.3

To prove Theorem 1.2 we will actually establish a stronger result. A path $P = x_1 x_2 \dots x_l$ in Q_n is an *increasing geodesic* if the directions of the edges $x_i x_{i+1}$ increase with i. An increasing geodesic P *ends* at a vertex x if $x = x_l$. For any vertex $x \in G$ we let $L_G(x)$ denote an increasing geodesic in G of maximum length which ends at x. The key idea to the proof is to show that on average $|L_G(x)|$ is large. This allows us to simultaneously keep track of geodesics for all vertices of G, which is vital in the inductive proof below.

Theorem 2.1. *Let G be a subgraph of Q_n of average degree d. Then*

$$\sum_{v \in V(G)} |L_G(v)| \geq d|G|.$$

Proof. Write $S(G)$ for $\sum_{v \in V(G)} |L_G(v)|$. We will show that for any subgraph G of Q_n, we have $S(G) \geq 2|E(G)|$, by induction on $|E(G)|$. The base case $|E(G)| = 0$ is immediate. Assume the result holds by induction for all graphs with $|E(G)| - 1$ edges and that we wish to prove the result for G.

Pick an edge $e = xy$ of G with largest coordinate direction and look at the graph $G' = G - e$. By the induction hypothesis, we have

$$S(G') = \sum_{v \in V(G')} |L_{G'}(v)| \geq 2|E(G')| = 2(|E(G)| - 1).$$

Now clearly we must have $|L_G(v)| \geq |L_{G'}(v)|$ for all vertices $v \in G$. Furthermore, notice that the coordinate direction of e cannot appear on the increasing geodesics $L_{G'}(x)$ and $L_{G'}(y)$. Indeed, the edge of $L_{G'}(x)$ adjacent to x has direction less than e and as $L_{G'}(x)$ is an increasing geodesic, the directions of all edges in $L_{G'}(x)$ must be less than e. We now consider two cases.

Case I: $|L_{G'}(x)| = |L_{G'}(y)|$. Then the paths $L_{G'}(x)xy$ and $L_{G'}(y)yx$ are increasing geodesics in G ending at y and x respectively. Therefore $|L_G(x)| \geq |L_{G'}(x)| + 1$ and $|L_G(y)| \geq |L_{G'}(y)| + 1$ and $S(G) \geq S(G') + 2 \geq 2|E(G')| + 2 = 2|E(G)|$.

Case II: $|L_{G'}(x)| \neq |L_{G'}(y)|$. Without loss of generality assume that $|L_{G'}(x)| \geq |L_{G'}(y)| + 1$. Then $L_{G'}(x)xy$ is an increasing geodesic ending at y of length $|L_{G'}(x)| + 1 \geq |L_{G'}(y)| + 2$. Therefore $|L_G(y)| \geq |L_{G'}(y)| + 2$ and $S(G) \geq S(G') + 2 \geq 2|E(G')| + 2 = 2|E(G)|$.

This concludes the inductive step and the proof. □

Note that it is immediate from Theorem 2.1 that $|L_G(v)| \geq d$ for some $v \in V(G)$ and therefore, G contains an increasing geodesic of length at least d, as claimed in Theorem 1.2.

We now give a strengthening of Theorem 2.1, showing that G must actually contain *many* geodesics of length d. First note that for $d \in \mathbb{N}$, taking a disjoint union of subgraphs isomorphic to Q_d gives a graph G with average degree d and exactly $d!|G|/2$ geodesics of length d. Indeed, suppose $G = \cup_i G_i$ where G_i are disjoint and isomorphic to Q_d for all i. Then any vertex in G_i is a starting vertex for $d!$ geodesics of length d. This gives $\sum_i d!|G_i|/2 = d!|G|/2$ geodesics in total. The following result proves that we can in fact guarantee this many geodesics of length d for general subgraphs of Q_n.

Theorem 2.2. *If G is a subgraph of Q_n with average degree at least d, then G contains at least $d!|G|/2$ geodesics of length d.*

Proof. We first use Theorem 2.1 to prove the following claim: G contains at least $|G|$ increasing geodesics of length d. To see this, first remove an edge e from G if it lies in at least two increasing geodesics of length d. Now repeat this with $G \setminus \{e\}$ and so on until we end up at a subgraph G' of G in which all edges lie in at most one increasing geodesic of length d. Let