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Hamiltonian cycles in Cayley graphs of imprimitive complex reflection groups

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ABSTRACT

Generalizing a result of Conway, Sloane, and Wilkes (1989) for real reflection groups, we show the Cayley graph of an imprimitive complex reflection group with respect to standard generating reflections has a Hamiltonian cycle. This is consistent with the long-standing conjecture that for every finite group, G, and every set of generators, S, of G the undirected Cayley graph of G with respect to S has a Hamiltonian cycle.

1. Introduction

For a finite group G and a subset S of $G \setminus \{1\}$, the (right, undirected) *Cayley graph* of G with respect to S, $\Gamma(G, S)$, has vertices corresponding to the elements $g \in G$ and edges (g, gs) and (g, gs^{-1}) for each $g \in G$ and $s \in S$. The Cayley graph is vertex-transitive, regular, and connected when S generates G, which we assume throughout. Label the edge from g to gs by s and that from gs to g by s^{-1} (note that edge labels for $\Gamma(G, S)$ are drawn from $S \cup S^{-1}$). It is common to consider both together as a single undirected edge with s and s^{-1} indicating travel along the edge in the appropriate direction.

A *path* in Γ is an ordered sequence of adjacent vertices in Γ and a path is *self-avoiding* if no vertex appears more than once. A *Hamiltonian path* is a self-avoiding path containing every vertex of Γ. When the initial and the final vertex of a Hamiltonian path are adjacent it determines a *Hamiltonian cycle* and a graph containing a Hamiltonian cycle is called *Hamiltonian*.

The question, dating back to 1969 in a monograph by Lovász, of whether every connected vertex-transitive graph has a Hamiltonian path, remains unresolved. The stronger claim, that every connected vertex-transitive graph has a Hamiltonian cycle, is known to be false and it has been observed that the four known counterexamples are not Cayley graphs. The resulting conjecture that for every finite group G and any generating set S the Cayley graph $\Gamma(G, S)$ has a Hamiltonian cycle also remains unresolved and finding such a cycle is an NP-complete problem in general. See [31,9,21,26] for surveys of the status and history of the problem and references, including those supplying counter-conjectures. When S is not closed under inversion, it is possible for the directed graph with vertices the elements of G and only edges (g, gs) for $g \in G$ and $s \in S$ to have no Hamiltonian cycle. For instance, the directed circulant graph on $\mathbb{Z}_{12}$ with generators 3, 4 (and 6) is not Hamiltonian (see [31,24]).

The conjecture that every (undirected) Cayley graph is Hamiltonian is easy to prove for abelian groups and known to be true for several specific types of groups that are nearly abelian, with either specific or arbitrary generating sets. For instance

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the conjecture has been shown true when

- G is a p-group [29],
- the commutator subgroup G' is a cyclic p-group [25,12,13,19,11],
- the order of G has few prime factors [22],
- the order of G is odd and G' has order pq or is cyclic of order $p^a q^b$ for $a, b \geq 0$ [30],
- G is nilpotent and G' is cyclic [15].

It is also known that the Cayley graph of the semidirect product of two cyclic groups with respect to a specific generating set is Hamiltonian [1]. We prove as our main result (Theorem 4.7) that the conjecture is true for the highly non-abelian infinite family of complex reflection groups, $G = G(de, e, n) \cong \mu^n \rtimes S_n$ with respect to commonly used generating sets of reflections. Here μ is the cyclic group of de-th roots of unity.

Main Result. *If G is an irreducible imprimitive complex reflection group and S is a standard generating set for G, then the (undirected right) Cayley graph $\Gamma(G, S)$ has a Hamiltonian cycle.*

Our result generalizes that in [7], which provides an algorithm to generate a Hamiltonian cycle in each $\Gamma(G, S)$ where G is a finite real reflection group and S is the standard set of generating simple reflections. That paper utilized the Coxeter presentation of the groups to give an inductive proof of the existence, and hence recursive construction, of a Hamiltonian cycle. It also explicitly treats the small number of base cases.

Although there is no such uniformly well-behaved presentation or set of generators for complex reflection groups (see [4,28]), we use those that go back to [8,6] and are given in the standard Refs. [5,23]. While these presentations, generating sets, and resulting Cayley graphs do not tend to satisfy the usual conditions for the existence of Hamiltonian cycles currently given in the literature (see further discussion at the end of Section 2), they do allow for an inductive approach similar to that in [7]. In order to exploit that approach we must treat six infinite families of groups as base cases. For three of these families we explicitly write down Hamiltonian cycles and in the remaining three cases our proofs provide a method for doing so.

In two cases we utilize a process we call flipping which is sometimes referred to in the literature as a Pósa exchange and is similar to a process utilized in a probabilistic algorithm to find Hamiltonian cycles in general graphs (see [2]). It would be interesting to further explore the application of the flipping process and related algorithms to Cayley graphs of complex reflection groups and in particular, to determine whether there exist obstructions to their success in these graphs and if so, under what conditions. Another interesting direction would be to investigate how our result might be of use in group coding (see [20]).

The paper is organized as follows. In Section 2, following [14], we review necessary facts about real reflection groups and summarize the classification of complex reflection groups due to Shephard and Todd [27]. The classification consists of a three-parameter infinite family, $G(de, e, n)$, along with 34 exceptional groups. This paper treats only the $G(de, e, n)$, though we have conducted some initial investigations for the exceptional complex reflection groups. Generating sets S for the $G(de, e, n)$ are also given in Section 2. Our formulations of the commonly used Factor Group Lemma (see [22]), the flipping process, and the method of lifting cycles from quotient graphs are described in Section 3. The main result and its proof, including all base case lemmas, appear in Section 4.

2. Background on reflection groups

A *Coxeter system* (G, S) is a group G with set of generators $S = \{s_1, \ldots, s_n\}$ that has a presentation of the form

$$G = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} \rangle,$$

where $m_{ii} = 1$ and $m_{ij} = m_{ji} \geq 2$ for $1 \leq i \neq j \leq n$. Such a group is called a *Coxeter group* and is more commonly denoted W due to the connection with Weyl groups. The presentations of Coxeter groups are classified using diagrams that graphically encode the s_i and the m_{ij}. The classification of finite irreducible Coxeter groups consists of four infinite families and six exceptional groups. Finite Coxeter groups have a geometric incarnation as they are exactly the finite groups generated by orthogonal reflections of a real vector space (see [3,17]). It can also be shown that there is a natural set of generating reflections up to conjugacy, so such a choice is fixed and these are termed *simple reflections*. Every Cayley graph $\Gamma(G, S)$ with G a finite irreducible Coxeter group and S a set of simple reflections is shown in [7] to have a Hamiltonian cycle.

The notions of reflection and classification of finite groups generated by reflections extend to the setting of an n-dimensional complex vector space V (for a brief survey, see [14]). A linear transformation $r : V \to V$ is a *reflection* if it is of finite order and has a $+1$-eigenspace of dimension $n - 1$. In the remaining complex dimension the reflection acts by a root of unity and hence may have order greater than two. A finite subgroup, G, of $\mathrm{GL}(V)$ generated by reflections is called a *reflection group on V*. Since G is finite, the standard averaging technique makes it possible to fix a non-degenerate G-invariant hermitian form on V and consider G as a subgroup of the unitary group on V. Finiteness of G also guarantees the representation on V is completely reducible, which means it suffices to consider reflection groups and spaces on which they act irreducibly. More precisely, G is said to act irreducibly in dimension k if its fixed point space is of dimension $n - k$ and it acts irreducibly when restricted to the complement of that fixed point space.