Discrete Mathematics 326 (2014) 66–74

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Enumeration of unlabeled uniform hypergraphs☆

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ARTICLE INFO

Article history:
Received 22 March 2013
Received in revised form 15 October 2013
Accepted 7 March 2014
Available online 25 March 2014

Keywords:
Unlabeled hypergraph
Uniform hypergraph
Enumeration

ABSTRACT

We consider the enumeration problem of unlabeled hypergraphs by using Pólya's counting theory and Burnside's counting lemma. Instead of characterizing the cycle index of the permutation group acting on the edge set $\mathcal{E}$, we treat each cycle in the cycle decomposition of a permutation ρ acting on $\mathcal{E}$ as an equivalence class (or transitive set) of $\mathcal{E}$ under the operation of the group generated by ρ. Compared to the cycle index-based method, our method is more effective to deal with the enumeration problem of hypergraphs. Using this method we establish an explicit counting formula for unlabeled k-uniform hypergraphs of order n, where k is an arbitrary integer with $1 \leq k \leq n-1$. Based on our counting formula, the asymptotic behavior for the number of unlabeled uniform hypergraphs is also analyzed.

1. Introduction

Graphical enumeration may date from 1857 when Cayley successfully found a recursive formula for the number of trees and rooted trees. Later in 1937, Pólya [14] developed a powerful theory for counting distinct arrangements of objects, which nowadays is known as Pólya's theorem or the Redfield–Pólya theorem. Theoretically, this theory provides us with a universal technique to count the unlabeled graphs, based on which the enumeration problems for various classes of graphs were studied in the literature, see for example [2,3,5,6,8,7,9,12,15] and the references cited therein. In particular, the pre-1973 work on graphical enumeration was nicely included in the text book [8] written by Harary and Palmer.

In this paper, we consider the enumeration problem of unlabeled simple hypergraphs. A *hypergraph* is defined as usual to be a pair $\langle V, \mathcal{E} \rangle$, where V is the vertex set and $\mathcal{E} \subseteq 2^V$ is the edge set. A hypergraph H is called uniform, or more specifically, *k-uniform* [17] with $1 \leq k \leq |V| - 1$ if each edge of H consists exactly of k vertices. In contrast, a hypergraph whose edges can consist of any number of vertices is called a *general* hypergraph. A hypergraph H is called *simple* if multiple edges are not allowed. In the following, all the hypergraphs are simple unless otherwise stated.

In contrast to unlabeled graphs, the enumeration of unlabeled hypergraphs, to a certain degree, was not much considered in the literature. The earlier work was done by de Bruijn and Klarner [4]. The counting formula for unlabeled general hypergraphs was deduced by Wu [19] and Ishihara [11] independently. Another interest is to enumerate the unlabeled hypergraphs with given number of vertices and edges in which multiple edges are allowed. The enumeration of such hypergraphs could be treated as that of a certain type of ordinary bipartite graphs, namely the bicolored graphs [7,13,16,18] (with only a slight difference when the two bipartite sets are of the same order) or equivalently, as that of the (0, 1)-matrices under row and column permutations [10].

We apply Pólya's counting theory and Burnside's counting lemma to our enumeration problem. In general, the key point of Pólya's theory to treat the number of cycles in the cycle decompositions of the permutations acting on the edge set is to

☆ Research supported by NSFC (11271307).
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http://dx.doi.org/10.1016/j.disc.2014.03.005
0012-365X/

characterize the cycle index of the permutation group, which has been widely used in graphical enumeration. However, in the enumeration of hypergraphs, to characterize the cycle index may become particularly complex [11,19], the usually used method for which is to use the generating function.

We will, however, not try to characterize the cycle index. Instead, we consider the number of cycles from a different point of view which arises from a simple observation, that is, a cycle in a permutation ρ acting on the edge set $\mathcal{E}$ could be regarded as an equivalence class of $\mathcal{E}$ under the operation of the group $\langle\rho\rangle$ generated by ρ. Therefore, the number of cycles is exactly that of the equivalence classes of $\mathcal{E}$ under the operation of $\langle\rho\rangle$.

Compared to the method based on the cycle index, our method is more effective to deal with the enumeration problem of hypergraphs, by which we deduce an explicit counting formula of unlabeled k-uniform hypergraphs for any integer k with $1 \le k \le |V| - 1$. In particular, our result yields a more succinct form of the well known counting formula for the ordinary graphs introduced by Harary and Palmer [8]. Using our method, we also simplify the counting formula of unlabeled general hypergraphs given by Wu [19] and Ishihara [11] obtained by using the generating function technique. Finally, based on our counting formulas, the asymptotic behaviors for the numbers of uniform and general hypergraphs are analyzed.

2. Counting formulas for unlabeled hypergraphs

Let ρ be a permutation of a finite set X. It is well known that ρ can be split in a unique way into cycles, that is, pairwise disjoint subsets of X that are cyclically permuted by ρ. We notice that any two elements in the same cycle of ρ can be permuted from one to another by ρ^t for some integer t. This means that we may treat each cycle of ρ, in terms of the standard Burnside's counting theory, as an equivalence class (or transitive set [1]) under the operation of the group generated by ρ, i.e., $\langle\rho\rangle = \{\rho, \rho^2, \dots, \rho^{|\rho|}\}$, where $|\rho|$ is the order of ρ, that is, the least positive integer l such that ρ^l is the identity permutation. Thus, the number of cycles of ρ, denoted by $\tau(\rho)$, is exactly the number of equivalence classes of X under the operation of $\langle\rho\rangle$. So by Burnside's lemma (Lemma 5.1, [1]), we have the following observation.

Observation 2.1.

$$\tau(\rho) = \frac{1}{|\rho|}\sum_{i=1}^{|\rho|}\psi(\rho^i),$$

where $\psi(\rho^i)$ is the number of elements of X that are invariant under ρ^i, that is, the number of $x \in X$ for which $\rho^i(x) = x$.

We now turn to the enumeration problem of hypergraphs. In the following, the vertex set of a hypergraph $H = \langle V, \mathcal{E}\rangle$ is always set to be $V = [n] = \{1, 2, \dots, n\}$. For a partition P of n, we write it either as the form $P : p_1 + p_2 + \cdots + p_q$ or as $P : 1\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n$, for the convenience of our discussion, where α_i is the number of the integers i in the partition. Given natural numbers $a_1, a_2, \dots, a_m$, we denote by $(a_1, a_2, \dots, a_m)$ and $[a_1, a_2, \dots, a_m]$ the greatest common divisor and the least common multiple of $a_1, a_2, \dots, a_m$, respectively.

A permutation $\pi \in S_n$ (the symmetry group on $[n]$) with cycle decomposition $\pi = \sigma_1\sigma_2\cdots\sigma_q$ induces a partition $P : p_1 + p_2 + \cdots + p_q$ of n, where p_i is the length of the cycle σ_i. Conversely, it is well known [8] that a partition $P : 1\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n$ of n induces

$$\frac{n!}{1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}\alpha_1!\alpha_2!\cdots\alpha_n!}$$

permutations in S_n, each of which has cycle decomposition of the form $1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}$, where i^{α_i} represents the product of α_i cycles of length i, $i \in \{1, 2, \dots, n\}$. For a permutation $\pi \in S_n$, we denote by $\rho(\pi)$ the permutation of the edge set $\mathcal{E}$ induced by π. Further, we use $\rho(P)$ to denote $\rho(\pi(P))$, where $\pi(P)$ is an arbitrary permutation in S_n induced by P.

Let $\mathcal{P}(n)$ be the class of all the partitions of n. In terms of Burnside's counting theory, we model a hypergraph H as an edge coloring of the complete hypergraph K_n of order n with two colors. Thus, by Burnside's lemma [1], the number of unlabeled hypergraphs of order n is given by

$$h(n) = \frac{1}{n!}\sum_{\pi\in S_n}\phi(\rho(\pi)) = \frac{1}{n!}\sum_{P\in\mathcal{P}(n)}\frac{n!}{1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}\alpha_1!\alpha_2!\cdots\alpha_n!}\phi(\rho(P)), \tag{2.1}$$

where $\phi(\rho(P))$ is the number of colorings that are invariant under $\rho(P)$. By Pólya's counting theory [1], we have $\phi(\rho(P)) = 2^{\tau(\rho(P))}$, where $\tau(\rho(P))$ is the number of cycles in $\rho(P)$. It can be seen that the order of the permutation $\rho(P)$ equals $[p_1, p_2, \dots, p_q]$. Thus, by Observation 2.1,

$$\tau(\rho(P)) = \frac{1}{[p_1, p_2, \dots, p_q]}\sum_{t=1}^{[p_1,p_2,\dots,p_q]}\psi(\rho^t(P)). \tag{2.2}$$

Let $\pi \in S_n$ be induced by a partition $P : p_1 + p_2 + \cdots + p_q$, with cycle decomposition

$$\pi = (n_{11}n_{12}\cdots n_{1p_1})(n_{21}n_{22}\cdots n_{2p_2})\cdots(n_{q1}n_{q2}\cdots n_{qp_q})$$