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Graphical cyclic permutation groups

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ABSTRACT

We establish conditions for a permutation group generated by a single permutation of a prime power order to be an automorphism group of a graph or an edge-colored graph. This corrects and generalizes the results of the two papers on cyclic permutation groups published in 1978 and 1981 by S. P. Mohanty, M. R. Sridharan, and S. K. Shukla.

1. Introduction

It is well known that while every abstract group is isomorphic to the automorphism group of a graph, not every permutation group can be represented directly as the automorphism group of a concrete graph.

The problem of representability of a permutation group $A = (A, V)$ as the full automorphism group of a graph $G = (V, E)$ was studied first for regular permutation groups. In particular, the question which abstract groups have a regular representation as an automorphism group of a graph (the so-called *GRR*) has received considerable attention. In the language of permutation groups this is the same as to ask which regular permutation groups are automorphism groups of the graphs. There were many partial results in this area (see for instance [10–12,17–19,21–23]). The full characterization has been obtained by Godsil [4] in 1979. In [1], L. Babai uses the result of Godsil to prove a similar characterization in the case of directed graphs.

In [15,16], S. P. Mohanty, M. R. Sridharan, and S. K. Shukla, consider cyclic permutation groups (i.e., generated by a single permutation) whose order is p^n for a prime p. In [16, Theorem 3], they describe all such groups for $p > 5$ that are automorphism groups of graphs. However, although the result is true, the proof contains an essential gap. Moreover, the authors make a false claim that there are no such groups for $p = 3$ or $p = 5$. We correct these results here and generalize as follows. First, we consider also the cases $p \in \{2, 3, 5\}$, which are different, and completely ignored in [15,16]. In fact the case $p > 5$ is the simplest and the least interesting. The difficulty and beauty of the problem is hidden in these three cases. At second, we generalize the results describing the cyclic permutation groups of a prime power order that are automorphism groups of *edge-colored graphs*. As in many earlier cases (see e.g. results in [20,7,6,5]) edge-colored graphs turn out to be a more appropriate setting for this kind of problems, giving simpler, and more elegant formulations of results.

In fact, the general problem has been considered already by H. Wielandt in [24]. Permutation groups that are automorphism groups of colored graphs were called 2^*-closed. In [14], A. Kisielewicz introduced the so-called *graphical complexity of permutation groups*. By $GR(k)$ we denote the class of automorphism groups of *k-colored graphs*, by which we mean the complete graphs whose edges are colored with at most k colors. By GR we denote the union of all classes $GR(k)$ (which is the class of all 2^*-closed groups). Moreover, we put $GR^*(k) = GR(k) \setminus GR(k-1)$, and for a permutation group A, we say that A has a *graphical complexity* k if $A \in GR^*(k)$. Then, $GR(2)$ is the class of automorphism groups of simple graphs.

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Now, the main natural problem in this subject is the characterization of permutation groups in *GR*, i.e., those permutation groups that are automorphism groups of k-colored graphs for any k. The most exciting question stated in [14] is however whether the hierarchy $GR(k)$ is a real hierarchy, that is, whether there are permutation groups in each of the classes $GR^*(k)$. For now we know only that $GR^*(k)$ are nonempty for all $k \leq 6$, which means in particular that there is a 6-colored graph whose automorphism group is different from all the automorphism groups of k-colored graphs with $k \leq 5$; such a graph on $n = 32$ vertices was constructed in [8].

There are various approaches to the topic. In [7] it is shown that the direct sum of two groups belonging to $GR(k)$ belongs itself to $GR(k+1)$, and most often it belongs to $GR(k)$. Similar results are obtained for other products. In [8,9], we have described all k-colored graphs with the highest degree of symmetry. In particular, we have shown that if a k-colored graph is edge-transitive and color-symmetric (i.e. there are permutations of the vertices, that permute colors of the edges in an arbitrary way), then $k \leq 5$.

In this paper we determine when a cyclic permutation group A of a prime power order belongs to *GR*, and show that if $A \in GR$, then $A \in GR(3)$. We also characterize, when it is in $GR(2)$, and when in $GR^*(3)$. In Section 2, we recall some definitions concerning edge-colored graphs and permutation groups. We also recall two results from [7], and prove their generalizations we need in the sequel. In Section 3, we complete the consideration of cyclic permutation groups of prime order started in [15,16]. In particular, we show that there are cyclic permutation groups of order 3 and 5 which belong to $GR(2)$, contradicting the claim given in [15]. In Section 4, we give a correct proof of the theorem stated in [16] concerning cyclic permutation group of prime power order p^n with $p > 5$, and complete the research considering the cases for $p = 2, 3, 5$.

2. Definitions and basic facts

We assume that the reader has the basic knowledge in the areas of graphs and permutation groups. Our terminology is standard and for nondefined notions the reader is referred to [2,25].

A *k-colored graph* (or more precisely *k-edge-colored graph*) is a pair $G = (V, E)$, where V is the set of vertices, and E is an *edge-color function* from the set $P_2(V)$ of two elements subsets of V into the set of colors $\{0, \ldots, k-1\}$ (in other words, G is a complete simple graph with each edge colored by one of k colors). In some situations it is helpful to treat the edges colored 0 as missing. In particular, a 2-colored graph can be treated as a usual simple graph. Generally, if no confusion can arise, we omit the adjective "colored". By a (sub)graph of G *spanned* by a subset $W \subseteq V$ we mean $G' = (W, E')$ with $E'(\{v, w\}) = E(\{v, w\})$, for all $v, w \in W$.

Let $v, w \in V$ and $i \in \{0, \ldots, k-1\}$. If $E(\{v, w\}) = i$, then we say that v and w are *i-neighbors*. In case when $k = 2$, we follow the terminology of simple graphs, and 1-neighbors are called simply *neighbors*. By $d_i(v)$ (*i-degree* of a vertex v) we denote the number of i-neighbors of v.

An automorphism of a colored graph G is a permutation σ of the set V of vertices preserving the edge function: $E(\{v, w\}) = E(\{\sigma(v), \sigma(w)\})$, for all $v, w \in V$. The group of automorphisms of G will be denoted by $Aut(G)$, and considered as a permutation group $(Aut(G), V)$ acting on the set of the vertices V.

Permutation groups are treated up to permutation isomorphism. Generally, a permutation group A acting on a set V is denoted (A, V) or just A, if the set V is clear or not important. By S_n, we denote the symmetric group on n elements, and by I_n the trivial one element group acting on n elements (consisting of the identity only). By C_n we denote a regular action of $\mathbb{Z}_n$. In particular, $S_2 = C_2$. Finally, D_n denotes the dihedral group of symmetries of n-cycle i.e., the group of automorphisms of a graph $G = (V, E)$ with $V = \{v_0, \ldots, v_{n-1}\}$, $E(\{v_i, v_{(i+1 \bmod n)}\}) = 1$ for all i, and $E(v_i, v_j) = 0$, otherwise. It is clear that C_n is a subgroup of D_n of index two. Note that since the complete graph K_1 on one vertex has no edges, $I_1 \in GR(0)$.

Later, we will use two kinds of products of permutation groups:

Direct sum. For permutation groups $A = (A, V)$, $B = (B, W)$ we consider the action of the direct product of groups $A \times B$ on the disjoint sum $V \cup W$ given by

$$(a, b)(x) = \begin{cases} a(x) & \text{for } x \in V, \\ b(x) & \text{for } x \in W. \end{cases}$$

The resulting group is called direct sum of A and B and denoted by $A \oplus B$.

Parallel product. For the permutation group (A, V), the parallel product $A^{(n)}$ is the permutation group $(A, V \times \{1, \ldots, n\})$ with the following natural action.

$$a((v_1, k)) = (a(v_1), k).$$

Now, we recall two theorems which are proved in [7] and will be used later.

Theorem 2.1 ([7, Corollary 3.5]). *Let $A = A_1 \oplus A_2$ be a direct sum of permutation groups. Then, $A \in GR$ if and only if each of A_1 and A_2 belongs to GR and A is not equal to $I_2 = I_1 \oplus I_1$.*

Theorem 2.2 ([7, Lemma 3.1 and Theorem 4.1]). *Let $A_1, A_2 \in GR(k)$, for some $k \geq 2$. Then,*

(1) $A_1 \oplus A_2 \in GR(k+1)$.
(2) *If $A_1 \neq A_2$, then $A_1 \oplus A_2 \in GR(k)$.*
(3) $A_1 \oplus I_n \in GR(k)$.