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## Note Note on the diameter of path-pairable graphs

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#### 1. Introduction

Given a fixed integer k, a graph G on at least 2k vertices is k-path-pairable if for any pair of disjoint sets of vertices  $X = \{x_1, \ldots, x_k\}$  and  $Y = \{y_1, \ldots, y_k\}$  of G there exist k edge-disjoint paths  $P_i$  such that  $P_i$  is a path from  $x_i$  to  $y_i$ ,  $1 \le i \le k$ . The path-pairability number of a graph G is the largest positive integer k for which G is k-path-pairable. The motivation of setting edge-disjoint paths between certain pairs of nodes naturally arose in the study of communication networks. There are various reasons to measure the capability of the network by its path-pairability number, that is, the maximum number of pairs of users for which the network can provide separated communication channels without data collision (see [1] for additional details). The nodes corresponding to the users are often called *terminal nodes* or *terminals*. A graph G on n = 2m

vertices is *path-pairable* if it is *m*-path-pairable, that is, for every pairing of the vertex set  $\{x_1, y_1, \ldots, x_m, y_m\}$  there exist edge-disjoint paths  $P_1, \ldots, P_m$  joining  $x_1$  to  $y_1, \ldots, x_m$  to  $y_m$ , respectively. By definition, path-pairable graphs are k-pathpairable for  $0 \le k \le \lfloor \frac{n}{2} \rfloor$ . The three dimensional cube  $Q_3$  and the Petersen graph P are both known to be path-pairable. The graph shown in Fig. 1 is the only path-pairable graph with maximum degree 3 on 12 or more vertices. Apart from such small and rather sporadic examples we only know few path-pairable families. Certainly, the complete graph  $K_{2k}$  on n = 2k vertices is path-pairable. It can be proved easily that the complete bipartite graph  $K_{a,b}$  is path-pairable as long as a + b is even and  $a, b \neq 2$ . Particular species of the latter family, the star graphs  $K_{2a+1,1}$  show that path-pairability is achievable even in the presence of vertices of small degrees. They also illustrate that vertices of large degrees are easily accessible transfer stations to carry out linking in the graphs without much effort. That motivates the study of k-path-pairable and path-pairable graphs with small maximum degree. Faudree, Gyárfás and Lehel [3] gave examples of k-path-pairable graphs for every  $k \in \mathbb{N}$  with maximum degree 3.

Note that their construction has exponential size in terms of k and is not path-pairable. Unlike in case of k-path-pairability, the maximum degree  $\Delta(G)$  must increase together with the size of a path-pairable graph G. Faudree, Gyárfás and Lehel [4] proved that if G is a path-pairable graph on n vertices with maximum degree  $\Delta$  then  $n \leq 2\Delta^{\Delta}$ . The theorem gives an approximate lower bound of  $\frac{\log(n)}{\log \log(n)}$  on  $\Delta(G)$ . By contrast, the graphs of the above presented families have maximum degree  $rac{n}{2}$  or more. Kubicka, Kubicki and Lehel [5] investigated path-pairability of complete grid graphs and proved that the twodimensional complete grid  $K_a \times K_b$  of size n = ab is path-pairable. For a = b that gives examples of path-pairable graphs

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A graph on 2k vertices is path-pairable if for any pairing of the vertices the pairs can be joined by edge-disjoint paths. The so far known families of path-pairable graphs have diameter of at most 3. In this paper we present an infinite family of path-pairable graphs with diameter  $d(G) = O(\sqrt{n})$  where *n* denotes the number of vertices of the graph. We prove that our example is extremal up to a constant factor.

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Fig. 1. Path-pairable graph of order 12.

with maximum degree  $\Delta = 2a - 2 < 2\sqrt{n}$ . In the same paper they raised the question about similar properties of threedimensional complete grids. Note that if path-pairable, the grid  $K_m \times K_m \times K_m$  yields an even better example of size  $n = m^3$ and maximum degree  $\Delta = 3(m - 1) = O(\sqrt[3]{n})$ .

We mention that one of the most interesting and promising path-pairable candidate is the *n*-dimensional hypercube  $Q_n$ .  $Q_1 = K_2$  is path-pairable while  $Q_2 = C_4$  is not as pairing of the nonadjacent vertices of the cycle cannot be linked. One can prove that the cube  $Q_3$  is path-pairable, the question for higher odd-dimensions has yet to be answered (it was proved in [2] that  $Q_n$  is not path-pairable for *n* even).

**Conjecture 1** ([1]). The (2k + 1)-dimensional hypercube  $Q_{2k+1}$  is path-pairable for all  $k \in \mathbb{N}$ .

A common attribute of the known path-pairable graphs is their small diameter. For each pair (x, y) of terminals in the examples above, the length of the shortest x, y paths is at most 3. While terminal pairs of an actual pairing may not always be joined by shortest paths, small diameter gives the advantage of quick accessibility of the vertices and makes designation of edge-disjoint paths easier. The question concerning the existence of an infinite family of path-pairable graphs with unbounded diameter naturally arises. We use the notation d(G) for the diameter of the graph G and  $d_{max}(n)$  for the maximal diameter of a path-pairable graph on n vertices. We mention that if true, Conjecture 1 proves the lower bound  $\log_2 n \le d_{max}(n)$  for  $n = 2^{2k+1}, k \in \mathbb{N}$ .

The main goal of this note is to study the largest possible diameter of path-pairable graphs. We present a family of pathpairable graphs  $\{G_n\}$  such that  $G_n$  has n vertices and diameter  $O(\sqrt{n})$  for infinitely many values of n. We show that our construction is optimal up to a constant factor by proving the following theorem.

**Theorem 1.** If *G* is a path-pairable graph on *n* vertices with diameter  $d \ge 20$  then  $d \le 6\sqrt{2} \cdot \sqrt{n}$ .

For a subgraph *H* of a graph *G*, |H| denotes the number of vertices in *H*. The degree of a vertex *x* and the distance of vertices *x* and *y* are denoted by d(x) and d(x, y), respectively. For additional details on path-pairable graphs we refer the reader to [1,2,4].

#### 2. Construction

We construct our example of a path-pairable graph on *n* vertices with diameter  $O(\sqrt{n})$  by the graph operation called "blowing-up". Let  $n = (2m) \cdot (4m + 3)$  and define *G* as an equally blown up graph of the cycle  $C_{2m}$  of size *n*, that is,  $V(G) = \{x_{i,j} : 0 \le i \le 2m - 1, 0 \le j \le 4m + 2\}$  and  $x_{i,j}$  and  $x_{i',j'}$  are connected if i - i' = 1 or i - i' = -1 (modulo 2m). We use the notation  $S_i = \{x_{i,j} \in V(G) : 0 \le j \le 4m + 2\}$  and refer to the set as the *i*th class of *G*. Easy to see that *G* has diameter  $m > \frac{1}{4}\sqrt{n}$  (in fact,  $d(G) \approx \frac{1}{2\sqrt{2}}\sqrt{n}$ ). We mention that *G* also has maximum degree  $O(\sqrt{n})$ , the same order of magnitude as in [5] which is the best known result for path-pairable graphs with small degree.

Set an arbitrary pairing of the vertices of *G*. We accomplish the linking of the pairs in two phases. During the first phase, for each pair of terminals, we define a path that starts at one of the terminals and ends at some vertex in the class of its pair. If the ending vertex happens to be the actual pair of the terminal, we set this path as the joining path for the given pair, otherwise we continue with the second phase. If two terminals initially belong to the same class, then the pair simply skips the first phase of the linking. Direct our cycle  $C_{2m}$  and the blown-up graph *G* counterclockwise and label each pair x, y such that there exists a directed  $x \rightarrow y$  path of length at most *m*. We start building the above mentioned path for pair (x, y) at vertex *x*. Fix *m* edge-disjoint matchings  $M_1^i, \ldots, M_m^i$  of size (4m + 3) between every consecutive classes  $S_i$  and  $S_{i+1}$ . For a pair of terminals (x, y) lying in classes  $S_i$  and  $S_{i+d}$  (modulo 2m) at distance d ( $1 \le d \le m$ ), choose the edge of  $M_1^i$  being adjacent to *x* and label the other vertex adjacent to it by  $p_1(x)$ . In step *j* for  $2 \le j \le d$  take the edge of  $M_j^{i+j-1}$  being adjacent to  $p_{j-1}(x)$  and label its other end by  $p_j(x)$ . Apparently, *y* and  $p_d(x)$  belong to the same class. Phase one ends by assigning a path  $P_{xy} = xp_1(x) \ldots p_d(x)$  to each (x, y) pair of terminals.

Observe that paths *P* and *P'* assigned to and starting at terminals *x* and *x'* of the same class do not contain a common vertex as they are given edges of the same matchings in every step. Now assume that edge  $e = (x_{i,j}, x_{i+1,k})$  has been utilized

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