

Nonleft peaks in Dyck paths: A combinatorial approach



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ABSTRACT

A peak in a Dyck path is called nonleft, if the ascent preceding it is greater than or equal to the descent following it. In this paper, we present a combinatorial construction of the set of Dyck paths of fixed semilength and number of nonleft peaks. As a bonus, we obtain various results on the enumeration of several kinds of peaks.

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1. Introduction

A Dyck path of semilength n is a lattice path consisting of n steps $u = (1, 1)$ (called *rises*) and n steps $d = (1, -1)$ (called *falls*), that starts and ends at the same height and lies weakly above this height. It is convenient to consider that the starting point of a Dyck path is the origin of a pair of axes (see Fig. 1).

The set of Dyck paths of semilength n is denoted by $\mathcal{D}_n$, and we set $\mathcal{D} = \bigcup_{n \geq 0} \mathcal{D}_n$, where $\mathcal{D}_0 = \{\varepsilon\}$ and ε is the empty path. Furthermore, we use the notation $\mathcal{D}^* = \mathcal{D} \setminus \{\varepsilon\}$. It is well known that $|\mathcal{D}_n| = C_n$, where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n th Catalan number; (sequence A000108 in [11]). We denote by

$$C = C(x) = \sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}$$

the generating function of the Catalan numbers, which is equivalently defined by the following equation:

$$C = 1 + xC^2, \quad C(0) = 1. \quad (1.1)$$

It is well known that the coefficients of the powers of C are given by the formula

$$C_n^{(s)} = [x^n]C^s = \frac{s}{n+s} \binom{2n+s-1}{n}, \quad s \in \mathbb{N}^*.$$

Every nonempty Dyck path α can be uniquely decomposed in the form $\alpha = u\beta d\gamma$, where $\beta, \gamma \in \mathcal{D}$. This is the so called *first return decomposition*. If $\gamma = \varepsilon$, then α is a *prime* Dyck path. Every Dyck path can be uniquely decomposed into prime paths, called *prime components*. For example, the prime components of the Dyck path in Fig. 1 are uudd and uuududddudd.

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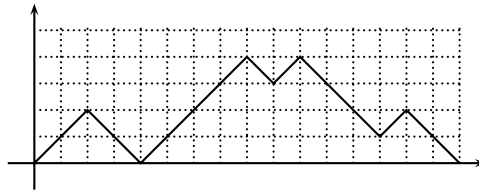


Fig. 1. The Dyck path uudduuuudddudd.

An *ascent* (resp. *descent*) in a Dyck path is a maximal sequence of consecutive rises (resp. falls). Every nonempty Dyck path α can be uniquely decomposed in the form $\alpha = u^h d \alpha_1 d \alpha_2 \cdots d \alpha_h$, where $h \geq 1$ and $\alpha_i \in \mathcal{D}$, $i \in [h]$. This is the so called *first ascent decomposition*.

A path $\tau \in \{u, d\}^*$, called in this context *string*, occurs in a path α if $\alpha = \beta \tau \gamma$, for some $\beta, \gamma \in \{u, d\}^*$. The number of occurrences of the string τ in α is denoted by $|\alpha|_\tau$. For example, for the path α of Fig. 1 we have that $|\alpha|_{ud} = 4$ and $|\alpha|_{uuu} = 2$. Many articles dealing with the occurrence of strings in Dyck paths have appeared in the literature (e.g., see [2,6–10,12]). More general results on this subject are given in [4,5].

A point P of a Dyck path α is called a *peak* if it is preceded by a rise and followed by a fall. Equivalently, a peak of α is an occurrence of the string ud in α . The statistic “number of peaks” in Dyck paths is enumerated (see [2]) by the Narayana numbers (sequence A001263 in [11]).

A peak P is called *right* (resp. *left*, resp. *centered*) if the ascent preceding P is longer than (resp. shorter than, resp. equal to) the descent following it. Notice that the first (resp. final) peak of each prime component of a Dyck path is nonleft (nonright).

The statistics “number of right (or left) peaks” and “number of centered peaks” in Dyck paths were presented by the authors in the 23rd British Combinatorial Conference (2011).

The statistic “number of nonleft peaks” was studied in an equivalent form in [1], where it was proved that the associated generating function $F = F(x, y)$ is the run transform of the Catalan generating function, satisfying the formula:

$$F = 1 + \frac{1-xy}{1-x} x F^2 + \frac{x(y-1)}{1-x} F. \quad (1.2)$$

In the following result, the coefficients of F are evaluated via the double sequence $(b_{n,k})$ of the Touchard numbers, counting the Dyck paths of semilength n , having k occurrences of the string duu (sequence A091894 in [11]):

$$b_{n,k} = 2^{n-2k-1} \binom{n-1}{2k} C_k, \quad b_{0,0} = 1.$$

Proposition 1. The number $a_{n,k}$ of all Dyck paths with n rises and k nonleft peaks is given by the formula:

$$a_{n,k} = [x^n y^k] F = \sum_{j=0}^{\lfloor \frac{n-k}{2} \rfloor} \binom{n-j-1}{n-k} b_{n-k,j}. \quad (1.3)$$

Formula (1.3) was suggested by Meijer in [11] (sequence A175136). An algebraic proof of this formula follows easily from the observation that

$$F(x, y) = 1 + \frac{xy}{1-xy} B\left(\frac{x}{1-xy}, xy\right),$$

where

$$B = B(x, y) = \sum_{\alpha \in \mathcal{D}} x^{|\alpha|_u} y^{|\alpha|_{duu}} = \sum_{n \geq 0} \sum_{k \geq 0} b_{n,k} x^n y^k$$

is the generating function which counts Dyck paths according to their semilength and number of duu 's, satisfying the equation (see [2])

$$B = 1 - (1-y)x + 2(1-y)xB + xyB^2.$$

We note that, applying a version of Lagrange inversion formula (see [2]) for the above equation, we obtain that the coefficients of B^i are given by the formula

$$[x^n y^k] B^i = \sum_{s=0}^{i-1} \binom{i}{s+1} b_{n,k,s}, \quad i \geq 1, \quad (1.4)$$

where

$$b_{n,k,s} = \binom{n-1}{2k+s} C_k^{(s+1)} 2^{n-2k-s-1}. \quad (1.5)$$

Clearly, the sequence $b_{n,k,s}$ is a generalization of the Touchard sequence, since $b_{n,k,0} = b_{n,k}$.

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