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Firefighting on square, hexagonal, and triangular grids

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ABSTRACT

In this paper, we consider the *firefighter problem* on a graph $G = (V, E)$ that is either finite or infinite. Suppose that a fire breaks out at a given vertex $v \in V$. In each subsequent time unit, a firefighter protects one vertex which is not yet on fire, and then the fire spreads to all unprotected neighbors of the vertices on fire. The objective of the firefighter is to save as many vertices as possible (if G is finite) or to stop the fire from spreading (for an infinite case).

The surviving rate $\rho(G)$ of a finite graph G is defined as the expected percentage of vertices that can be saved when a fire breaks out at a vertex of G that is selected uniformly random. For a finite square grid $P_n \square P_n$, we show that $5/8 + o(1) \leq \rho(P_n \square P_n) \leq 67\,243/105\,300 + o(1)$ (leaving the gap smaller than 0.0136) and conjecture that the surviving rate is asymptotic to $5/8$.

We define the surviving rate for infinite graphs and prove it to be $1/4$ for the infinite square grid, even for more than one (but finitely many) initial fires. For the infinite hexagonal grid we provide a winning strategy if two additional vertices can be protected at any point of the process, and we conjecture that the firefighter has no strategy to stop the fire without additional help. We also show how the speed of the spreading fire can be reduced by a constant multiplicative factor. For triangular grid, we show that two firefighters can slow down the fire in the same sense, which is relevant to the conjecture that two firefighters cannot contain the fire on the triangular grid, and also corrects a previous result of Fogarty (2003).

1. Introduction

The following *firefighter problem* on a graph $G = (V, E)$ was introduced by Hartnell at a conference in 1995 [9]. Suppose that a fire breaks out at a given vertex $v \in V$. In each subsequent time unit (called a turn), a firefighter *protects* one vertex which is not yet on fire and then the fire spreads to all unprotected neighbors of the vertices already on fire. Once a vertex is on fire or is protected it stays in such state forever. Protecting a vertex is in essence equivalent to deleting it from the graph.

The game stops if no neighbor of the vertices on fire is unprotected and the fire cannot spread. If the graph is finite, the game finishes at some point and the goal of the firefighter is to save as many vertices as possible. In case of an infinite graph, the goal of the firefighter is to stop the fire from spreading or, if this is not possible, to save as many vertices as possible in the limit (we introduce this graph parameter in Section 3).

Today, almost 20 years later, our knowledge about this problem is much greater and a number of papers have been published. We would like to refer the reader to the survey of Finbow and MacGillivray for more information [6].

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For finite graphs, we focus on the following property. Let $\mathrm{sn}(G, v)$ denote the number of vertices in G the firefighter can save when the fire breaks out at a vertex $v \in V$, assuming the best strategy is used. Then let $\rho(G, v) = \mathrm{sn}(G, v)/n$ be the proportion of vertices saved (here and throughout the paper, n denotes the number of vertices of G, assuming G is finite). The *surviving rate* $\rho(G)$ of G, introduced in [4], is defined as the expected $\rho(G, v)$ when the fire breaks out at a random vertex v of G (uniform distribution is used), that is,

$$\rho(G) = \frac{1}{n}\sum_{v\in V}\rho(G, v) = \frac{1}{n^2}\sum_{v\in V}\mathrm{sn}(G, v).$$

For example, it is not difficult to see that for cliques $\rho(K_n) = \frac{1}{n}$, since no matter where the fire breaks out only one vertex can be saved. For paths we get that

$$\rho(P_n) = \frac{1}{n^2}\sum_{v\in V}\mathrm{sn}(G, v) = \frac{1}{n^2}\left(2(n-1) + (n-2)(n-2)\right) = 1 - \frac{2}{n} + \frac{2}{n^2}$$

(one can save all but one vertex when the fire breaks out at one of the leaves; otherwise two vertices are burned).

It is not surprising that almost all vertices on a path can be saved, and in fact, all trees have this property. Cai, Cheng, Verbin, and Zhou [1] proved that the greedy strategy of Hartnell and Li [10] for trees saves at least $1 - \Theta(\log n/n)$ percentage of vertices on average for an n-vertex tree. Moreover, they managed to prove that for every outer-planar graph G, $\rho(G) \geq 1-\Theta(\log n/n)$. Both results are asymptotically tight and improved upon earlier results of Cai and Wang [2]. (Note that there is no hope for a similar result for planar graphs, since, for example, $\rho(K_{2,n}) = 2/(n+2) = o(1)$.) However, this does not mean that it is easy to find the exact value of $\rho(G)$. It is known that the decision version of the firefighter problem is NP-complete even for trees of maximum degree three [5].

Moving to another interesting direction, the third author of this paper showed that any graph G with average degree strictly smaller than 30/11 has the surviving rate bounded away from zero [12] and showed that this result is sharp (the construction uses a mixture of deterministic and random graphs). (See [13] for a generalization of this result for the k-many firefighter problem.) These results improved earlier observations of Finbow, Wang, and Wang [7].

1.1. Our contribution

First, we study the surviving rate of $P_n\Box P_n$, the Cartesian product of two paths of length $n-1$. It was announced by Cai and Wang that

$$0.625 + o(1) = \frac{5}{8} + o(1) \leq \rho(P_n\Box P_n) \leq \frac{37}{48} + o(1) \approx 0.7708$$

but a formal proof has not been published. We will prove the following result, which provides much better upper bound.

Theorem 1. *For the Cartesian product of two paths we have*

$$0.625 + o(1) = \frac{5}{8} + o(1) \leq \rho(P_n\Box P_n) \leq \frac{67243}{105300} + o(1) < 0.6386.$$

Our proof for the upper bound is not very sophisticated and there are ways to improve it. On the other hand, it narrows down the surviving rate to a small interval smaller than 0.0136. It is natural to conjecture the following but this still remains open.

Conjecture 2. $\lim_{n\to\infty}\rho(P_n\Box P_n) = 5/8$.

For an infinite graph $G = (V, E)$, the primary goal is to determine if the fire can be stopped from spreading or not. All graphs we discuss here are vertex transitive so the choice of the starting point is irrelevant.

It is known (and easy to show) that it is impossible to surround the fire with one firefighter in the infinite Cartesian grid (see [14,8]). On the other hand, it is clear that two firefighters can stop the fire (that is when two vertices can be protected in each round) and in [14] the optimal strategy was provided that does it in 8 steps. (See [3] for a fractional version of this problem.) It was proved in [8] that if the fire breaks out on the triangular grid, three firefighters contain the fire easily, but the proof that two firefighters cannot contain the fire is unfortunately flawed (as we discuss below) and this question is still open.

For the infinite square grid $G_\Box$, we show that it is optimal to save a 90° wedge of vertices. In Section 3 we formally introduce a measure of the surviving rate for infinite graphs and show that $\rho(G_\Box) = 1/4$.

For the infinite hexagonal grid G_{hex}, we show that one firefighter can save 2/3 of the grid, and with just a little additional help of two extra protected vertices, it is possible to stop the fire from spreading.

Theorem 3. $\rho(G_{hex}) \geq 2/3$.

Moreover, if the firefighter is allowed to protect one extra vertex at time t_1 and one at time t_2, $1 \leq t_1 \leq t_2$ (possibly with $t_1 = t_2$), the firefighter will contain the fire on G_{hex}. Moreover, the strategy does not need to know t_1 and t_2 in advance.