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On 1-Hamilton-connected claw-free graphs

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ABSTRACT

A graph G is k-Hamilton-connected (k-hamiltonian) if $G-X$ is Hamilton-connected (hamiltonian) for every set $X \subset V(G)$ with $|X| = k$. In the paper, we prove that

(i) every 5-connected claw-free graph with minimum degree at least 6 is 1-Hamilton-connected,
(ii) every 4-connected claw-free hourglass-free graph is 1-Hamilton-connected.

As a byproduct, we also show that every 5-connected line graph with minimum degree at least 6 is 3-hamiltonian.

1. Introduction

We follow the most common graph-theoretical terminology and for concepts and notations not defined here we refer e.g. to [1]. Specifically, by a *graph* we mean a finite undirected graph $G = (V(G), E(G))$; in general, we allow graphs to have multiple edges. We use $d_G(x)$ to denote the *degree* of a vertex x, and we set $V_i(G) = \{x \in V(G)|d_G(x) = i\}$, $V_{\leq i}(G) = \{x \in V(G)|d_G(x) \leq i\}$ and $V_{\geq i}(G) = \{x \in V(G)|d_G(x) \geq i\}$. The *weight* of an edge e is the number of edges incident with e and distinct from it.

For a set $M \subset V(G)$, $\langle M\rangle_G$ denotes the *induced subgraph* on M, and for a simple graph F, G is said to be F-*free* if G is a simple graph that does not contain an induced subgraph isomorphic to F. Specifically, for $F = K_{1,3}$ we say that G is *claw-free*. The *hourglass* Γ is the only graph with degree sequence 4, 2, 2, 2, 2 (see Fig. 2), and for $F = \Gamma$ we say that G is *hourglass-free*.

The *neighborhood* of a vertex x, denoted $N_G(x)$, is the set of all neighbors of x, and a vertex $x \in V(G)$ is *simplicial* (*locally connected, locally disconnected, eligible*) if $\langle N_G(x)\rangle_G$ is a complete (connected, disconnected, connected noncomplete) subgraph of G. We will use $V_{EL}(G)$ to denote the set of all eligible vertices in G. The *closed neighborhood* of a vertex x is the set $N_G[x] = N_G(x) \cup \{x\}$, and an edge $e \in E(G)$ is *pendant* if one of its vertices is of degree 1.

For $x \in V(G)$, $G - x$ is the graph obtained from G by removing x and all edges incident to it. If $x, y \in V(G)$ are such that $e = xy \notin E(G)$, then $G + e$ is the graph with $V(G + e) = V(G)$ and $E(G + e) = E(G) \cup \{e\}$, and, conversely, for $e = xy \in E(G)$ we denote $G - e$ the graph with $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$. Specifically, for $F \subset G$ and $e \in E(G)$, we set $F - e = F$ if $e \in E(G) \setminus E(F)$. We use $\omega(G)$ to denote the *number of components* of G.

A graph G is *hamiltonian* if G contains a *hamiltonian cycle*, i.e. a cycle of length $|V(G)|$, and G is *Hamilton-connected* if, for any $a, b \in V(G)$, G contains a *hamiltonian* (a, b)-*path*, i.e., an (a, b)-path P with $V(P) = V(G)$. For $k \geq 1$, G is k-*hamiltonian* if $G - X$ is hamiltonian for every set of vertices $X \subset V(G)$ with $|X| = k$, and k-*Hamilton-connected* if $G - X$ is Hamilton-connected for every set of vertices $X \subset V(G)$ with $|X| = k$. Note that a hamiltonian graph is necessarily 2-connected, a

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Hamilton-connected graph must be 3-connected, a k-hamiltonian graph must be $(k+2)$-connected, and if G is k-Hamilton-connected, then G must be $(k+3)$-connected. The *line graph* of a graph H is the simple graph $L(H)$ with vertex set $E(H)$, in which two vertices are adjacent if and only if the corresponding edges of H share a vertex, and a graph G is a *line graph* if there is a graph H such that $G = L(H)$. Note that every line graph is claw-free, and that the degree of a vertex in G equals the weight of the corresponding edge in H.

The main motivation of our research is the following conjecture by Thomassen.

Conjecture A ([18]). *Every* 4*-connected line graph is hamiltonian.*

There are many known equivalent versions of the conjecture (see [3] for a survey on this topic). We mention here the following one, which is of importance for our results.

Theorem B ([16]). *The following statements are equivalent:*

(i) *Every* 4*-connected line graph is hamiltonian.*
(ii) *Every* 4*-connected claw-free graph is* 1*-Hamilton-connected.*

In this paper, we prove the following two results giving a partial affirmative answer to the statement (ii) of Theorem B, i.e., equivalently, to Conjecture A:

- in Section 3, we show that every 4-connected claw-free hourglass-free graph is 1-Hamilton-connected,
- in Section 4, we show that every 5-connected claw-free graph with minimum degree at least 6 is 1-Hamilton-connected.

As a byproduct, in Section 5 we show that every 5-connected line graph with minimum degree at least 6 is 3-hamiltonian.

2. Preliminaries

In this section we summarize some background knowledge that will be needed for our results.

Let H be a graph and $G = L(H)$. It is well-known that if we allow H to be a multigraph, then (unlike in line graphs of simple graphs), for a line graph G, a graph H such that $G = L(H)$ is not uniquely determined. A simple example is the hourglass Γ in Fig. 2, where Γ is the line graph of all three graphs to the right. As shown in [14], this difficulty can be overcome by imposing an additional requirement that simplicial vertices in G correspond to pendant edges in H.

The *basic graph* of a multigraph is the simple graph with the same vertex set, in which every multiedge is replaced by a single edge. A *multitriangle* (*multistar*) is a multigraph such that its basic graph is a triangle (star). The *center* of a multistar S with m edges is the vertex $x \in V(S)$ with $d_S(x) = m$ (for $|V(S)| = 2$ we choose the center arbitrarily), and all other vertices of S are its *leaves*. An induced multistar S in H is *pendant* if none of its leaves has a neighbor in $V(H) \setminus V(S)$, and similarly a multitriangle T is pendant if exactly one of its vertices (called the *root*) has neighbors in $V(H) \setminus V(S)$. We will use the following operations, introduced in [20] (Operation B) and [14] (Operation C).

Operation B. Choose a pendant multitriangle in H with vertices $\{v, x, y\}$ and root v, delete all edges joining v and x, and add the same number of edges between v and y.

Operation C. Choose a pendant multistar in H and replace every leaf of degree $k \geq 2$ by k leaves of degree 1.

Now, for a multigraph H, $BC(H)$ denotes the multigraph obtained from H by recursively repeating operations B and C. It can be shown that $L(H) = L(BC(H))$, $BC(H)$ is uniquely determined and has the property that simplicial vertices in $L(H)$ correspond to pendant edges in H.

Proposition C ([14]). *Let G be a connected line graph of a multigraph. Then there is, up to an isomorphism, a uniquely determined multigraph H such that a vertex $e \in V(G)$ is simplicial in G if and only if the corresponding edge $e \in E(H)$ is a pendant edge in H.*

For a line graph G, we will always consider its preimage to be the unique multigraph with the properties given in Proposition C; this preimage will be denoted $L^{-1}(G)$. Similarly, we will write $x = L(e)$ and $e = L^{-1}(x)$ if $e \in E(L^{-1}(G))$ is the edge corresponding to a vertex $x \in V(G)$. In accordance with our definitions, when working with a claw-free graph or with a line graph G, we always consider G to be a simple graph, while if G is a line graph, for its preimage $H = L^{-1}(G)$ we always admit H to be a multigraph, i.e. we always (even if we say "a graph H") allow H to have multiple edges.

An edge cut R of a graph H is *essential* if $H - R$ has at least two nontrivial components. For an integer $k > 0$, H is *essentially k-edge-connected* if every essential edge cut R of G contains at least k edges. Obviously, a line graph $G = L(H)$ of order at least $k + 1$ is k-connected if and only if the graph H is essentially k-edge-connected.

Given a trail T and an edge e in a multigraph G, we say that e is *dominated (internally dominated)* by T if e is incident to a vertex (to an interior vertex) of T, respectively. A trail T in G is called an *internally dominating trail*, shortly *IDT*, if T internally dominates all the edges in G. For $e_1, e_2 \in E(G)$, an IDT with terminal edges e_1, e_2 will be referred to as an (e_1, e_2)*-IDT*. If T is a closed trail, every vertex of T is considered to be an internal vertex, hence every dominated edge is internally dominated, and we simply say that T is a *dominating trail.* A classical result by Harary and Nash-Williams [5] shows that a line graph $G = L(H)$ of order at least 3 is hamiltonian if and only if H contains a dominating closed trail. The following result relates hamiltonian paths in a line graph to internally dominating trails in its preimage.